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COMPLETE SCHOOL ALGEBRA

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PREFACE

The "New Complete School Algebra" is intended for the use of the same classes as was the earlier "Complete School Algebra, Revised Edition." The present book combines the material of the "New First Course in Algebra" and the "New Second Course in Algebra" in a form convenient for use in courses of more than a year in length.

A large amount of review material is to be found at the point where the second year of work commences, since it was felt that there should be an adequate opportunity for the pupil to return to the fundamentals of algebraic technique before proceeding with the more advanced work of the second year.

As in the case of the former edition, emphasis has been placed on the Oral Exercises which appear at the beginning of the treatment of each new topic. The hint has also been used freely in the lists of written exercises in order to call the attention of the pupil to the best approach to the particular problem represented in the new exercises, before he has had a chance to develop wrong habits in the performance of his task.

The work on graphs has been considerably modified in treatment to conform to modern tendencies in graphical presentation. The first approach to the subject is from the direction of graphing statistics. The pupil will probably already be familiar with graphs as they appear in the daily newspapers and current periodicals. This work is included early in the text as it is of interest to the student and tends to make him feel more at home in the subject of algebra.

The next approach to graphs is from a more purely mathematical angle. The graphs of equations and functions are used to bring out the idea of the variable. The method of plotting the graphs of linear equations, as well as quadratic and higher-degree equations, is discussed and explained. The entire presentation of the subject, as well as the accompanying exercises, is intended not only to be of practical value to the pupil but to stimulate his interest in the study of the algebraic expression.

Throughout the book every effort has been made to provide the pupil with problems of real interest dealing with situations already more or less familiar to him. In doing this, equal care has been taken to make certain that such scientific and engineering formulas as are given shall be accurate and of real use in the fields from which they have been taken. Thus the pupil will be less likely to receive a distorted picture of the type of problem in which the applied scientist is interested, and will therefore gain familiarity with the way in which the formula is used in scientific work. This familiarity should be of value to him in his future work in scientific subjects.

The chapter on Linear Systems does not include solution by determinants, as it was felt that the inclusion of this topic at this point might confuse the pupil and divert his attention from points more essential to a clear understanding of the elements of the subject. But since there are many teachers who do wish to give their classes a knowledge of the solution of linear systems by the method of determinants, a separate chapter on Determinants and their application in the solution of equations has been included.

The chapters on Logarithms and Trigonometry have been placed together and somewhat earlier in the book than was formerly the case, as these topics are of interest and value to the pupil in fields other than algebra.

The authors wish to take this opportunity to express their appreciation to the many teachers in all parts of the country, whose criticism and suggestions have been of such great assistance in the preparation and revision of this series.

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CHAPTER I

INTRODUCTION

1. Numbers and symbols in arithmetic. In arithmetic, integers or integral numbers, such as 5, 18, 203, etc., are used to indicate the number of objects or units in a given group which have been or which can be counted. In each case these symbols for integers consist exclusively of the Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

When in the study of arithmetic one's attention is called to less than or more than all the parts of one object, the amount considered is represented by a fractional number, such as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{3}$, 0.8, and 1.25.

The fundamental processes of arithmetic included addition, subtraction, multiplication, and division, which were indicated by the symbols $+$, $-$, $\times$, and $\div$ respectively.

The number symbols and the symbols of operation with numbers which were used in arithmetic will be used in algebra as well. One of the striking features which will appear as the student enters upon his work in algebra will be the extent to which additional symbols are used to represent numbers.

2. Number symbols of algebra. In arithmetic, letters are rarely used for numbers, but in algebra, letters as well as Arabic numerals are used to represent numbers.

Thus, in the statement *interest* = *principal* × *rate*, i may be used to represent the number of dollars in the interest paid, p, the number of dollars in the principal or sum borrowed, and r, the rate of interest. Then the fact that interest = principal × rate is expressed in terms of these letters by

$$\boxed{i = p \times r}$$

The practice of representing numbers by letters illustrates one of the main features of algebra as distinguished from arithmetic. By this means we are frequently able to condense into a compact form a statement which is much longer and often less clear when expressed verbally.

3. Symbols of multiplication in algebra. In addition to the symbol ×, which is used in both arithmetic and algebra to indicate the process of multiplication, frequent use is made in algebra of the dot (·) placed between two number symbols, as in $a \cdot b$, to indicate the multiplication of a and b. The product of numbers is also indicated by writing the number symbols in immediate succession, as in $2a$, $5x$, cd, etc.

Hence the product of five and t may be expressed as $5 \times t$, as $5 \cdot t$, or as $5t$, of which the last method is the most common in algebra.

A number multiplied by itself is said to be *squared*, and the resulting product is called the *square* of the original number. The symbol for the operation of squaring a number is a small 2 placed at the upper right of the number squared. Thus 2 squared $= 2 \cdot 2 = 2^2 = 4$.

ORAL EXERCISES

Express the following in words:

1. 3×4.
2. $5 \times a$.
3. $2 \cdot s$.
4. $7c$.
5. $s + 5$.
6. $c - 6$.
7. $2m + 3$.
8. $a \div 2$.
9. $4 \times t = 8$.
10. $3f = 8$.
11. $c - 5 = 8$.
12. $5x + 2 = 12$.
13. $3n - n = 18$.
14. $p \cdot r = i$.
15. $A = lw$.
16. $d = rt$.
17. $p = 2l + 2w$.
18. $A = \frac{1}{2} \times ab$.
19. $A = s^2$.
20. $N = s^2h$.
21. $A = \frac{22}{7} r^2$.
22. $r = \frac{d}{t}$.
23. $d = \frac{1}{2} gt^2$.
24. $v = lwh$.
25. $V = \frac{22}{7} r^2h$.

Origin of symbols. Many of the symbols that are in common use in algebra at the present time have histories which not only are interesting in themselves but which also serve to indicate the slow and uncertain development of the subject. It is often found that symbols which seem without meaning represent some abbreviation or suggestion long since forgotten, and that operations and methods which we find hard to master have sometimes required hundreds of years to perfect.

In the early centuries there were practically no algebraic symbols in common use; one wrote out in full the words *plus*, *minus*, *equals*, and the like. But in the sixteenth century several Italian mathematicians used the initial letters *p* and *m* for + and −. Some think that our modern symbol − came into use through writing the initial *m* so rapidly that the curves of the letter gradually flattened out, leaving finally a straight line. The symbol + may have originated similarly in the rapid writing of the letter *p*. In the opinion of others these symbols were first used in the German warehouses of the fifteenth century to mark the weights of boxes of goods. If a lot of boxes, each supposed to weigh 100 pounds, came to the warehouse, the weight would be checked, and if a certain box exceeded the standard weight by 5 pounds, it was marked $100 + 5$; if it lacked 5 pounds, it was marked $100 - 5$. Though the first book to use these symbols was published in 1489, it was not until about 1630 that they could be said to be in common use.

Both the symbols for multiplication given in the text were first used about 1630. The cross was used by two Englishmen, Oughtred and Harriot, and was probably an adaptation of the letter x, which is found some years earlier. The dot is first found in the writings of the Frenchman Descartes. It is interesting to note that Harriot was sent to America in 1585 by Sir Walter Raleigh, and returned to England with a report of observations. He made the first survey of Virginia and North Carolina, and constructed maps of those regions.

It is strange that the line was used to denote division long before any of the other symbols here mentioned were in use. This is, in fact, one of the oldest signs of operation that we have. The Arabs, as early as A.D. 1000, used both $\frac{a}{b}$ and a/b to denote the quotient of a and b. The symbol $\div$ did not occur until about 1630.

Equality has been denoted in a variety of ways. The word *equals* was usually written out in full until about the year 1600, though the two sides of an equation were written one over the other by the Hindus as early as the twelfth century. The modern sign $=$ was probably introduced by the Englishman Recorde, in 1557, because, he says, "Noe. 2. thynges can be moare equalle" than two parallel lines. This symbol was not generally accepted at first, and in its place the symbols ||, $\propto$, and ∞ are frequently met during the next fifty years.

EXERCISES

Use symbols to indicate the following numbers and operations with numbers:

1. Three increased by two.
2. Twelve decreased by five.
3. A less six.
4. C increased by 8.

5. Two times a.
6. One increased by 5 times x.
7. M added to 7.
8. 2 plus a.
9. Four minus 2 times y.
10. Area of a rectangle equals length times width.
11. Area of a triangle equals one half the product of the base and altitude.
12. Area of a circle equals $\frac{22}{7}$ times the square of its radius.
13. Distance equals speed times the time.
14. Perimeter of a rectangle equals twice the length plus twice the width.
15. Average of two numbers equals their sum divided by 2.
16. Area of a square equals the square of its side.
17. Volume of a rectangular solid equals the product of the length, breadth, and thickness.
18. Volume of a cylinder equals the area of its base multiplied by its altitude.
19. A certain number divided by 3 is 7 with a remainder of 2.

4. Algebraic expressions. A letter or a group of number symbols involving one or more letters which stand for numbers is called an *algebraic expression.*

Thus, if n stands for some number, $7n$, $n + 2$, $5n - 1$, $\frac{n}{2}$, etc. are algebraic expressions.

Algebraic expressions in which the letters stand for the measures of distances, volumes, intervals of time, or other quantities are often called *formulas.*

Since each letter in an algebraic expression or formula stands for some number, it follows that the entire expression stands for some number and depends for its value on the values of the several letters involved.

Thus, if $a = 2$, then the algebraic expression $a + 5$ stands for 7; but if $a = 9$, then $a + 5$ stands for 14.

Again, if $m = 4$ and $n = 6$, then mn stands for 24, and $mn - 2$ stands for 22; but if $m = 5$ and $n = 7$, then mn stands for 35 and $mn - 2$ stands for 33.

ORAL EXERCISES

1. What number does $3d$ represent if $d = 10$? if $d = 100$?

2. What number does $5x + 2$ represent if $x = 1$? if $x = 3$?

3. What number does $2d - 7$ represent if $d = 4$? if $d = 10$?

4. What number does $l + w$ represent if $l = 5$ and $w = 2$? if $l = 10$ and $w = 8$?

5. What number does $a + b + c$ represent if $a = 12$, $b = 5$, and $c = 9$? if $a = 8$, $b = 15$, and $c = 10$?

6. The fact that p, the distance around a square, is four times the length of one of the sides, s, is stated by the expression $p = 4s$. Find p, if $s = 3$ inches; if $s = 5$ inches; if $s = 12$ inches; if $s = 15$ feet; if $s = 21\frac{1}{2}$ feet.

7. The fact that the perimeter of a rectangle is the sum of twice the length increased by twice the width is expressed by $p = 2l + 2w$. Find the perimeter of a rectangle which is 4 inches long and 3 inches wide. Find p if $l = 4$ inches and $w = 6$ inches; if $l = 9$ inches and $w = 8$ inches; if $l = 10$ feet and $w = 1$ foot; if $l = 18$ feet and $w = 1\frac{1}{2}$ feet.

8. A classroom is 28 feet by 36 feet. Find its perimeter. If it were 1 foot longer and 5 feet wider what would be its perimeter?

9. If p is the perimeter of a triangle and the sides are a, b, and c respectively, then $p = a + b + c$. Find the perimeter of a triangle whose sides are 3 inches, 4 inches, and 6 inches respectively. Find p if $a = 10$ inches, $b = 5$ inches, and $c = 12$ inches; if $a = 2$ feet, $b = 5$ feet, and $c = 3\frac{1}{2}$ feet.

10. The fact that the area of a rectangle is the product of the length and width is expressed by $A = lw$, where these dimensions are expressed in the same unit of measure. Find the area of a rectangle which has a length of 8 inches and a width of 5 inches. Find A if $l = 10$ inches and $w = 4$ inches; if $l = 4$ feet and $w = 2$ feet 6 inches; if $l = 20$ rods and $w = 40$ rods; if $l = 6$ feet and $w = 27$ inches.

11. Since the interest earned equals the product of the principal invested and the rate paid, then $i = pr$. Find i if $p = \$400$ and $r = 0.06$.

12. Compute the corresponding values of i for the given values of p and r in $i = pr$ and complete the following table:

$$i = pr$$

If $p =$	\$200	\$100	\$150	\$275	\$620	\$1220	\$2000
and $r =$	5%	6%	5%	7%	$6\frac{1}{2}$%	$4\frac{1}{4}$%	$4\frac{3}{4}$%
then $i =$	\$10	?	?	?	?	?	?

13. The fact that the area of a square equals the square of its side is expressed by $A = s \times s$ or $A = s^2$. Complete the following table, giving the correct values for A for each of the several values of s:

$$A = s^2$$

If $s =$	3 in.	5 in.	$3\frac{1}{2}$ in.	12 in.	2 ft.	5 yd.	$7\frac{2}{3}$ rd.	50 mi.
then $A =$	9 sq. in.	?	?	?	?	?	?	?

14. The fact that the volume of a rectangular solid, such as a chalk box, a block of ice, or a rectangular room, is the product of the length, width, and height of the solid is stated as $V = lwh$. Compute the volumes of the rectangular solids for the given values of the dimensions and complete the following table:

$$V = lwh$$

If $l =$	3 in.	5 in.	7 in.	6 in.	5 ft.	8 ft.	3 ft.
and $w =$	8 in.	6 in.	10 in.	12 in.	8 ft.	$2\frac{1}{2}$ ft.	10 in.
and $h =$	2 in.	4 in.	12 in.	$1\frac{1}{2}$ in.	2 ft.	4 ft.	15 in.
then $V =$	48 cu. in.	?	?	?	?	?	?

15. The approximate distance which a falling body travels through in a given number of seconds is expressed by $d = \frac{1}{2}at^2$, in which $d =$ the distance fallen in feet, a has the constant value 32, and $t =$ the time in seconds. Find the distances fallen for each of the time intervals given below:

$$d = \frac{1}{2}at^2$$

a, =	32	32	32	32	32	32	32
t, in seconds =	2	3	5	1	6	10	20
d, in feet =	64	?	?	?	?	?	?

5. Number representation through algebraic symbols. It has been shown that the value of an algebraic expression which involves one or more letters depends upon the values assigned to these letters. The fact that a given expression, such as $l \cdot w$, can represent the area of any rectangle or that $p \cdot r$ can represent any interest payment illustrates one of the principal advantages of algebraic over arithmetic methods. The manner of expressing one number in terms of one or more other number symbols is illustrated below.

Example

Write the expression for a number which is 2 greater than a given number, n.

Solution. **Let** $n =$ **the given number.**

Then $n + 2 =$ **the required number.**

EXERCISES

Write an algebraic expression for a number which is

1. 3 greater than b.
2. 5 less than r.
3. r less than 5.
4. Twice as great as b.
5. Five times as great as r.
6. One half as great as x.
7. Two thirds as great as x.
8. Three times d.
9. Equal to the product of h and b.
10. One half of the product of a and b.
11. Equal to the quotient of b divided by 5.
12. Five less than 3 h.
13. Two greater than the product of m and n.
14. Twelve less than the quotient of x divided by f.
15. Equal to the sum of ten times a and three times b.
16. Equal to the area of a rectangle whose base is b and whose altitude is a.
17. Equal to the area of a square whose side is s; whose side is x.

18. Equal to the cubic units in the volume of a box whose length is l, whose width is w, and whose height is h; whose length is 10, whose width is w, and whose height is h.

19. The speed of one automobile is two thirds that of another. Represent the speed of the slower if the faster moves 24 miles per hour; r miles per hour.

20. A boy is five years older than his brother. If b represents the age of the boy, represent the age of the brother.

21. A child deposits p cents in a savings bank each week. Another child deposits each week five cents less than the first child. How many cents does the second child deposit weekly?

22. If x represents a given number, what will represent a number one greater than x? two greater than x? three greater than x?

Even integers are those exactly divisible by 2. Odd integers are those not exactly divisible by 2.

Consecutive integers are integers arranged in the natural order, like 4, 5, 6, 7, 8, etc.

Consecutive odd integers are odd integers arranged in the natural order, like 5, 7, 9, 11, 13, etc.

Consecutive even integers are even integers arranged in the natural order, like 4, 6, 8, 10, 12, etc.

23. What is the difference between any two consecutive integers? between any two consecutive odd integers? between any two consecutive even integers?

24. If n is an integer, what is the next consecutive integer? If x is an integer, what are the next two consecutive integers?

25. If m is an odd integer, what is the next consecutive odd integer? the next two consecutive even integers?

26. If x is an odd integer, is $x + 1$ odd or even? If x is even, what of $x + 1$?

27. If x is an odd integer, what of $x + 3$? $x + 6$? If y is even, what of $y + 2$? $y + 3$?

28. If a is an integer, is $2a$ odd or even? Is $2a + 1$ odd or even?

29. Is the sum of two consecutive integers always odd?

HINT. Express the integers algebraically, add them, and divide by 2.

30. Is the sum of three consecutive integers always even?

31. By what number is the sum of three consecutive integers always divisible?

32. Write four consecutive integers the first of which is n; $n + 2$; $n + 7$; four the last of which is $n + 12$; $n - 6$; $n - 3$.

33. Write a series of three consecutive even numbers. If n stands for an even number, what will represent the next two consecutive even numbers?

34. If n represents an odd number, what will represent the next two consecutive odd numbers?

35. If a boy is 17 years old at present, how old was he 2 years ago? n years ago? How old will he be 5 years hence? n years hence?

36. If a boy is y years old and his sister is twice as old as he, how old will she be 3 years hence? 7 years hence? How old was she three years ago? n years ago?

37. If n represents the number of nickels in a collection, what will represent their value in cents? their value in dimes? in dollars?

38. If q represents a number of quarters in a certain collection, what will represent their value in cents? in nickels? in half dollars?

39. If a collection is composed of d dimes and h half dollars, represent the value of the collection in cents; in nickels; in dimes; in dollars.

40. How much change should be received from a five-dollar bill in payment for an article costing \$2? n dollars?

6. Simplifying algebraic expressions. In the same way that it is possible to add two number symbols such as 5 and 3, just so it is possible to add two or more number symbols which involve the same literal parts.

Thus $5t + 2t = 7t$

and $3a + 4a - 1a = 6a$

and $2x + 5 + x + 2 = 3x + 7.$

EXERCISES

Simplify the following by collecting the like number symbols:

1. $5 + 2.$
2. $8 - 5.$
3. $4s + 2s.$
4. $5h - 3h.$
5. $5r - 2r + 3.$
6. $2h + 10h - 2.$
7. $12h + 10h - 2.$
8. $3 + 2a + 5a - 2.$
9. $5x - 6x + 3x.$
10. $2 - m + 7 + 5m.$
11. $5a + b - 3a.$
12. $3x - 2y - 7y - x.$
13. $4m + 7n - 2n.$
14. $3a + 2a = 10.$
15. $5r - 3r = 8.$
16. $12x - 5x = 18 + 3.$
17. $4h + 7h = 33 - 11.$
18. $6r - 5r = 1 + 8.$
19. $k + 8k = 20 - 2.$
20. $s - 8s + 12s = 32 - 7.$

7. Solving equations. A statement of equality between two equal numbers or number symbols is called an *equation.*

Thus $3f = 36$ is an equation, for it is obviously true when $f = 12$.

Also $5t + 2 = 12$ is an equation, for it is true when $t = 2$.

The process of determining the particular value of the letter for which the statement is true is called *solving the equation.* Skill and accuracy in solving equations are fundamental to the study of algebra.

EXAMPLES

1. Solve for d, $10d = 250.$

Solution. $10d = 250.$

Dividing by 10, $d = 25.$

Check. Substituting 25 for d in $10d = 250$,

$$10 \times 25 = 250,$$

or $$250 = 250.$$

2. Solve for m, $2m + 3m = 35 + 5.$

Solution. $2m + 3m = 35 + 5.$ (1)

Collecting, $5m = 40.$

Dividing by 5, $m = 8.$

Check. Substituting 8 for m in (1),

$$2 \times 8 + 3 \times 8 = 35 + 5,$$

or $$40 = 40.$$

EXERCISES

Solve each of the following equations for the number represented by the letter involved:

1. $2n = 10.$
2. $3n = 21.$
3. $5n = 20.$
4. $4a = 20 + 8.$
5. $3x = 35 - 2.$
6. $7r = 47 + 2.$
7. $9s = 15 + 30.$
8. $12m = 38 - 2.$
9. $8n = 60 - 4.$
10. $15p = 50 - 5.$
11. $2n + 3n = 15.$
12. $5r - 2r = 30.$

13. $3h + 7h = 100$.
14. $12h - 5h = 56$.
15. $3m - m = 12$.
16. $2b + 3b = 6 + 9$.
17. $5x - 3x = 12 - 4$.
18. $x + x + 3x = 2 + 8$.
19. $h + 3h + 2h = 48$.
20. $5m - m + 3m = 70 - 7$.

8. Use of symbols in problem solving. By means of algebraic symbols and equations involving number symbols a simple and direct method is provided for the solution of certain types of problems.

In order that problems may be expressed in form suitable for algebraic treatment, it is often necessary to translate a verbal statement containing number relationships into an expression involving algebraic symbols. The ability to do this is one of the most important results of the study of algebra.

EXAMPLE

Find two numbers such that one is four times the other and their sum is 110.

Solution. Greater number + less number = 110.

Let $l =$ the less number.

Then $4l =$ the greater number.

$$4l + l = 110.$$

$$5l = 110.$$

Dividing by 5, $l = \frac{110}{5} = 22$, the less number.

Then $4l = 4 \times 22 = 88$, the greater number.

Check. Testing in the conditions set by the problem,

$$88 = 4 \times 22$$

and $88 + 22 = 110$.

PROBLEMS

1. The sum of two numbers is 72. One number is 5 times the other. Find the numbers.

2. Two boys together have 40 dollars. The first has 7 times as much as the second. How many has each?

3. The first of three numbers is twice the second, and the second is twice the third. Their sum is 77. What are the numbers?

HINT. Let x equal the third number.

4. Three boys together own 50 chickens. The first has four times as many as the third, and the second has five times as many as the third. How many has each?

5. Three newsboys together sold 108 papers. The first sold three times as many as the second, who sold twice as many as the third. How many did each boy sell?

6. What is the area of a triangle whose base is twice as long as the altitude, and the sum of whose base and altitude is 18 inches?

HINT. $A = \frac{1}{2}$ base × altitude.

7. The perimeter of a rectangle is 200 feet. The length is four times the width. Find the dimensions.

8. The perimeter of a rectangle is 96 feet. It is 7 times as long as it is wide. Find the dimensions.

9. The perimeter of a certain square is 116 inches. What is the length of one side?

10. What is the side of a square whose perimeter is 256 feet? of one whose perimeter is 320 feet?

11. The perimeter of a rectangle formed by placing two equal squares side by side is 72 inches. Find the side of each square.

12. The perimeter of a rectangle formed as in Problem 11 is 432 inches. What is the side of the square?

13. A rectangle is three times as long as it is wide. Its area is 147 square feet. Find its dimensions.

14. There are four numbers whose sum is 288. The second number is twice the first, the third number is 3 times the first, and the fourth is 3 times the second. What are the numbers?

15. The perimeter of a certain rectangle is 460 feet. It is 4 times as long as it is wide. Find its dimensions.

16. Three equal squares are placed side by side; the perimeter of the rectangle thus formed is 64 feet. What are the dimensions of the squares?

17. Four equal squares are placed together so as to form another square whose perimeter is 96 feet. What is the side of each square?

9. Terms. Algebraic expressions are often regarded as made up of parts separated by the signs of operation + or −. Each of these parts is called a *term*.

Thus, $3n$ is an expression of one term; $n + 3$, of two terms; $na + 2b - c$, of three terms, etc. Each of these expressions is an algebraic symbol for a number.

10. Factors. A *factor* of a product is any one of the number symbols which when multiplied together form the product.

Thus $2mn$ means 2 times m times n. Here 2, m, and n are each factors of $2mn$.

Again, $4(l + w)$ means 4 times the sum of l and w. Here the number 4 and $l + w$ are factors of the expression.

EXERCISES

1. Name the factors in 3×5; in $2 \times 7 \times 11$; in $5\,ab$; in lwh; in $\frac{1}{2}\,ba$; in $\frac{1}{3}\,Ba$; in $a(x+r)$; in $h(b+a)$.

Find the value of each of the following expressions for the following values of the factors: $a = 6$, $b = 2$, $n = 7$, $l = 4$, $w = 2$, and $x = 9$.

2. $5\,a$. 3. $3\,n$. 4. ab. 5. lw. 6. $\frac{1}{2}\,ba$. 7. nab.

8. $\frac{a}{w}$.

9. $\frac{l}{w} + 1$.

10. $5nlw + 3$.

11. $2\,l + 2\,w$.

12. $ba \div 2$.

13. $a + b + w$.

14. $3\,a - n + x$.

15. $a - b + n - 1$.

16. $ab - n + wx$.

17. $3\,x + w - l + 1$.

18. $4\,w - l + 3\,x - b$.

19. $\frac{a}{b} + \frac{l}{w}$.

20. $\frac{a}{2} - \frac{x}{3}$.

11. Exponents. An *exponent* is a positive integer written at the right and a little above another number to indicate how many times the number is to be used as a factor. (Later this definition will be extended to include fractions and other numbers as exponents.)

Thus, 3^2 means 3×3; 2^4 means $2 \times 2 \times 2 \times 2$; b^3 means $b \times b \times b$; and $3\,s^2$ means $3 \times s \times s$.

If a number is used once as a factor its exponent is 1, as in $2\,a^1$. This exponent is usually not written.

ORAL EXERCISES

1. What operations are indicated in 5^2? in 3^4? in 3×7^2? in 3×5^3? in $3^2 \times 2^4$?

2. Name the exponents used in the several terms of Exercise 1.

3. Name the exponents in $2\,a^2$, $5\,x^3$, $3\,c$, $2\,m^2n^3$; in $3\,x^2 \div 1$, $ab \times 2$, s^2h.

In the following substitute 2 for m and 5 for n and find the value of each expression:

4. m^2.
5. $m^2 + 1$.
6. $m^2 + n$.
7. $m^2 + n^2$.
8. $2\ n^3$.
9. $m^3 + 5$.
10. $25 + n^2$.
11. m^2n^2.
12. $m^2n + 1$.
13. $3\ mn^2 - 2$.
14. $5 + m^2 + n^2$.
15. $m^2 + mn + n^2$.
16. $3 - m + n^2$.
17. $n^2 - m^2 - nm$.
18. $m^3 + n^3 - 3$.

12. Coefficients. If a number is the product of two or more factors, either of these factors is called the *coefficient* of the product of the others.

Thus, in 5 × 3, 5 is the coefficient of 3, and 3 is the coefficient of 5. In $3\ a^2x$, 3 is the coefficient of a^2x, x is the coefficient of $3\ a^2$, and a^2 is the coefficient of $3\ x$.

The numerical coefficient 1, as in $1\ a$, $1\ mn$, etc., is usually omitted, but is understood.

ORAL EXERCISES

Find the value of the following:

1. $4\ h^2$ if $h = 5$; if $h = 3$.
2. hk^2 if $h = 2$ and $k = 3$.
3. $5\ m^3$ if $m = 2$; if $m = 1$; if $m = 5$.
4. rs^3 if $r = 2$ and $s = 5$.
5. $2\ lw^2$ if $l = 5$ and $w = 4$.
6. $3\ a^2b$ if $a = 2$ and $b = 3$; if $a = 5$ and $b = 2$.
7. Read the numerical coefficients in the terms of the preceding.
8. Read the exponents of each factor in the terms of the preceding.

13. Parentheses and radical signs. If two or more terms connected by signs of operation are inclosed in parenthesis, the entire expression is treated as a single number symbol.

Thus $2(5+3)$ means 2×8, or 16; $(6+3) \times (5-2)$ means 9×3, or 27; $(7-1) \div 2$ means $6 \div 2$, or 3; $(2+5)^2$ means 7^2, or 49; $3(a^2+b^2)$ means 3 times the sum of a^2 and b^2.

As in arithmetic, the symbol for square root is $\sqrt{\ }$, and that for cube root is $\sqrt[3]{\ }$.

The name *radical sign* is applied to all symbols like the following: $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$. The small figure in the radical sign, like 3 in $\sqrt[3]{\ }$, is called the *index* of the radical.

ORAL EXERCISES

Find the value of

1. $3(2+5)$.
2. $5(6-1)$.
3. $(2+1)(3+4)$.
4. $(3+2)(9-7)$.
5. $(9-3) \div 3$.
6. $(12-8) \div (6-4)$.
7. $(a+x)^2$ if $a=6$ and $x=1$.
8. $(m-n)^3$ if $m=9$ and $n=4$.
9. $\sqrt{9}+\sqrt[3]{8}$.
10. $\sqrt{6+3}$.
11. $2^2+\sqrt{4}$.
12. $6+3^2$.
13. $4^2-\sqrt{4}$.
14. $\sqrt{3^2+4^2}$.
15. $\sqrt[3]{5^2+2}$.
16. $\sqrt{(2+5)(8-1)}$.
17. $\sqrt[3]{3(2+1)(7-4)}$.
18. $\sqrt{n}+x-y$ if $n=25$, $x=4$, and $y=6$.
19. $(\sqrt{x}-3)+y$ if $x=49$ and $y=2$.

NOTE. There has been a considerable variety in the symbols for the roots of numbers. The symbol $\sqrt{\ }$ was introduced in 1544 by the German, Stifel, and is a corruption of the initial letter of the Latin word *radix*, which means "root." Before his time square root was denoted by the symbol ℞, used nowadays by physicians on prescriptions as an abbreviation for the word recipe. Thus $\sqrt[4]{5}$ would have been denoted by ℞^{4}5. Some early writers used a dot to indicate square root, and expressed $\sqrt{2}$ by $\cdot\, 2$.

14. Order of fundamental operations. In evaluating such an expression as $36 \div 3 - 2 \times 6$, the numbers 12, 10, and a final result of 60 are obtained if the operations are performed in the order in which they occur. If, however, the indicated division and multiplication are performed before the indicated subtraction, the result is 12 minus 12, and a final result of 0 is obtained. These results show that the value of numeric expressions may depend on the order in which the indicated operations are performed. It is customary to observe the following

RULE. *In a series of operations involving addition, subtraction, multiplication, and division of arithmetical numbers, the multiplications and divisions shall be performed first in the order in which they occur.*

The additions and subtractions in the resulting expression shall then be performed in the order in which they occur or in any other order.

If parentheses occur, each expression within a parenthesis should first be simplified in accordance with the preceding rule and the rule then applied to the entire expression.

EXAMPLES

Simplify:

1. $18 \div 2 + 5 - 4 \times 2.$

Solution. $18 \div 2 + 5 - 4 \times 2 =$

$$9 + 5 - 8 = 6.$$

2. $24 \div 8 \cdot 2 - 4(7 - 3) \div 8 + 2(5 + 6 \cdot 3).$

Solution. $24 \div 8 \cdot 2 - 4(7 - 3) \div 8 + 2(5 + 6 \cdot 3) =$

$$6 - \qquad 2 + \qquad 46 \qquad = 50.$$

EXERCISES

Simplify the following:

1. $20 - 5 + 6 - 8$.
2. $16 + (8 + 3)$.
3. $12 - (5 + 2)$.
4. $9 - (7 - 3)$.
5. $14 - (16 - 7) + 12 - 5$.
6. $6 \div 2 + 2$.
7. $8 \cdot 6 \div 4 - 10$.
8. $24(2 + 3)$.
9. $23 - 2 \cdot 6 - 8 \div (5 - 1)$.
10. $(10 - 3) \cdot (16 - 3 \cdot 3 + 8 \div 2)$.
11. $(14 - 3)(16 \div 4 \cdot 5) + 8 \cdot 2$.
12. $(16 - 6)(17 - 7) \div 100 \cdot 5 + 13$.
13. $3^2 + 2 \cdot 3 + 2^2$.
14. $8^2 - 2 \cdot 7 \cdot 3 + 3^3$.
15. $3 \cdot 4 - 6 \cdot 0 + 3 \times 5 \div 2^2 \times 7 - 3^2$.
16. $2^2 \cdot 3^2 - 2 \times 2 \times 3 \times 5 + 5^2$.

Find the value of

17. $(a + b)^2$ if $a = 2$ and $b = 3$; if $a = 5$ and $b = 2$.
18. $(x - y)^2$ if $x = 5$ and $y = 2$; if $x = 6$ and $y = 4$.
19. $A^2 - 2A - 8$ if $A = 10$; if $A = 8$; if $A = 12$.
20. $m^2 + m - 110$ if $m = 12$; if $m = 10$.
21. $x^2 + (a - b)^2$ if $x = 8$, $a = 10$, and $b = 6$.
22. $a^2 + \sqrt{b^2 + c^2}$ if $a = 6$, $b = 3$, and $c = 4$.
23. $(h - k)^2 - \sqrt{h^2 - k^2}$ if $h = 5$ and $k = 3$.

CHAPTER II

GRAPHICAL REPRESENTATION

15. Use of graphs. One often finds it difficult to grasp the important facts indicated by a column of figures, even though the meaning of each individual number is perfectly understood. When, however, these numbers are presented in the form of a diagram or picture, the facts frequently stand out in a clear and striking manner. Such a pictorial representation made for the purpose of comparing several similar or related quantities is called a *graph.*

Today one finds graphs in newspapers, in popular magazines, in scientific journals, and in social, industrial, and business reports. The ability to grasp quickly the important points presented by a graph — that is, the ability to comprehend a graph — is necessary for all who would read with understanding. Furthermore, the ability to present statistical data, whether of a scientific or a commercial nature, effectively by means of a graph is often very desirable.

16. Types of graphs. There are many ways of comparing quantities by means of graphs, but only three types will be considered here. These three types illustrate the most important methods of graphic presentation and comparison. They are:

1. The *bar graph,* in which the comparison is shown by parallel bars of the same width, but with lengths proportional to the quantities represented.

2. The *broken-line graph,* drawn through points located at distances from the base line proportional to the quantities they represent.

3. The *circle graph,* in which the relative sizes of the quantities compared are represented by sectors of the same circle.

These methods will be treated in the order mentioned above.

17. The bar graph. In this device the several numbers or quantities to be compared are represented by bars or heavy lines. This type of graph is well adapted to the comparison of quantities or magnitudes which are similar but which are not necessarily related to one another.

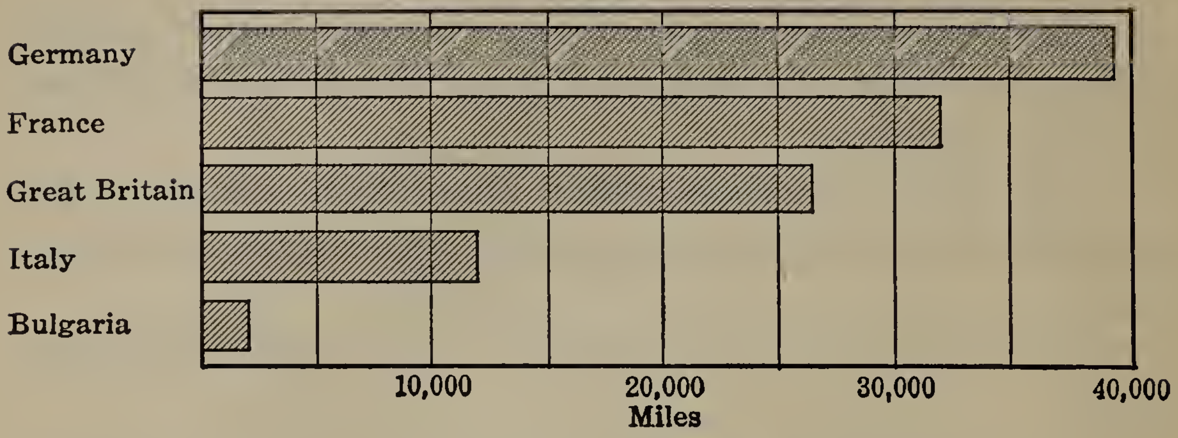
EXAMPLE

The railroad mileage in each of several European countries, to the nearest 500 miles, is as follows:

Germany	39,500 miles
France	32,000 miles
Great Britain	26,500 miles
Italy	12,000 miles
Bulgaria	1,500 miles

Indicate the relationship between these values by means of the bar graph.

Solution. It will be convenient to let one of the horizontal divisions represent the length of five thousand miles of railway. Then we can draw a bar for Germany 7.9 units in length, for France 6.4 units, for Great Britain 5.3 units, for Italy 2.4 units, and for Bulgaria .3 unit in length. If we make each of the bars one half unit in width we shall get the graph shown on the following page.



EXERCISES

Use bar graphs to show the following relationships:

1. The basins of the following rivers contain the indicated number of square miles to the nearest 100,000:

Mississippi and tributaries	1,300,000
Yukon	300,000
Columbia	300,000
Amazon	2,500,000
Nile	1,100,000

2. The naval expenditures to the nearest $1,000,000 of the Great Powers for the year 1921–1922 were as follows:

Great Britain	$406,000,000
United States	426,000,000
France	182,000,000
Italy	81,000,000
Japan	249,000,000

3. In 1922 various crops were produced in the United States in the following amounts to the nearest 100 million bushels:

Corn	3700 million bushels
Wheat	3000 million bushels
Oats	3000 million bushels
Barley	1000 million bushels
Rye	800 million bushels

4. The average monthly production of cement to the nearest 100 thousand barrels in the United States for a period of years was as follows:

YEAR	THOUSANDS OF BARRELS	YEAR	THOUSANDS OF BARRELS
1913	7,700	1919	6,700
1914	7,400	1920	8,300
1915	7,100	1921	8,200
1916	7,600	1922	9,500
1917	7,700	1923	11,500
1918	5,900		

QUERY. How did the slump in the building trades, which lasted from 1918 to 1920, affect the cement production in the United States?

18. The broken-line graph. This type of graph is particularly effective for representing the relation between two quantities or magnitudes. This type of graph is frequently used for the representation of changes in some quantity taking place during some particular interval of time.

EXAMPLE

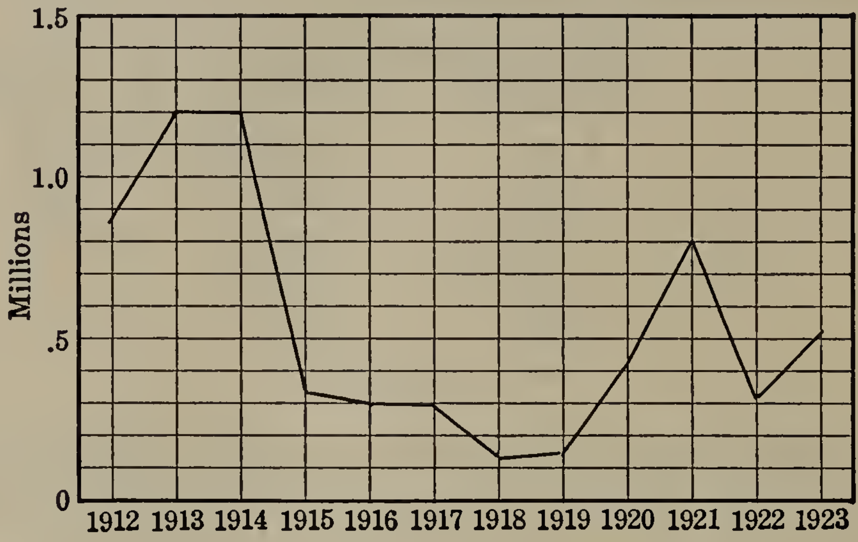
Plot the data contained in the following table:

NUMBER OF IMMIGRANTS ARRIVING IN UNITED STATES, 1912–1923

1912	840,000	1918	110,000
1913	1,200,000	1919	140,000
1914	1,220,000	1920	430,000
1915	330,000	1921	810,000
1916	300,000	1922	310,000
1917	300,000	1923	520,000

Solution. Let one space along the base line represent an interval of one year and let one space along the vertical line at the left represent 100,000 immigrants.

Then the following graph shows the trend of immigration over the period:



QUERIES. What effect did the outbreak of the European war apparently have on the immigration into the United States?

Did the entry of the United States into the war have any noticeable effect on the curve?

EXERCISES

Use broken-line graphs to show the following data:

1. The number of motor cars to the nearest 10,000 (both passenger cars and trucks) registered in the United States on January 1 of each of several years was as follows:

YEAR	NUMBER OF CARS	YEAR	NUMBER OF CARS
1915	2,450,000	1920	9,230,000
1916	3,510,000	1921	10,470,000
1917	4,980,000	1922	12,240,000
1918	6,150,000	1923	13,000,000
1919	7,570,000	1924	15,280,000

QUERIES. Assuming that the trend during the next year is the same as that over the past several years, what will be the approximate number of cars in the United States in 1925?

Did the World War have any effect on the registration of cars?

2. The population of Philadelphia to the nearest 1000 for several census dates was as follows:

YEAR	POPULATION	YEAR	POPULATION
1860	566,000	1900	1,294,000
1870	674,000	1910	1,549,000
1880	847,000	1920	1,824,000
1890	1,047,000		

3. The cotton production in the United States to the nearest 100,000 bales for a ten-year period was as follows:

YEAR	THOUSANDS OF BALES	YEAR	THOUSANDS OF BALES
1912	13,700	1917	11,300
1913	14,100	1918	12,000
1914	16,100	1919	11,400
1915	11,200	1920	13,400
1916	11,500	1921	8,300

4. The number, to the nearest 100, of secondary schools in the United States reporting to the commissioner of education was as follows:

YEAR	1890–1891	1900–1901	1910–1911	1919–1920
Secondary schools in United States	4500	8200	12,200	16,400

5. Enrollments in the secondary schools of the United States were as follows:

YEAR	1890–1891	1900–1901	1910–1911	1919–1920
Number of pupils	310,000	650,000	1,115,000	2,041,000

QUERY. From the trends shown in the graph, estimate the approximate number of pupils there will be in the public secondary schools in 1930.

6. The depth of a river was measured every five feet from bank to bank, giving the following records: 0, 1 foot, 2 feet 6 inches, 4 feet, 7 feet 9 inches, 10 feet, 15 feet, 16 feet 6 inches, 16 feet, 17 feet, 18 feet 8 inches, 17 feet 3 inches, 15 feet, 14 feet 6 inches, 12 feet, 10 feet 9 inches, 10 feet, 8 feet 3 inches, 6 feet, 5 feet 9 inches, 3 feet, 2 feet 9 inches, 1 foot 3 inches, 0. Plot, using a line graph.

QUERY. If these quantities were plotted as distances below the horizontal axis, what would the appearance of the curve suggest?

7. The enrollments, to the nearest 1000, reported for California secondary schools for a period of years were as follows:

SCHOOL YEAR	NUMBER OF PUPILS	SCHOOL YEAR	NUMBER OF PUPILS
1912–1913	58,000	1917–1918	125,000
1913–1914	66,000	1918–1919	137,000
1914–1915	76,000	1919–1920	162,000
1915–1916	95,000	1920–1921	191,000
1916–1917	113,000	1921–1922	225,000

8. The gold coin held in the United States during a given period was as follows:

YEAR	VALUE IN MILLIONS OF DOLLARS	YEAR	VALUE IN MILLIONS OF DOLLARS
1913	1870	1919	3110
1914	1890	1920	2710
1915	1990	1921	3300
1916	2450	1922	3790
1917	3020	1923	4210
1918	3080		

QUERY. What effects of the following occurrences are indicated on this curve? (1) the outbreak of the World War; (2) the entrance of the United States into the war; (3) the close of hostilities in 1918; (4) the industrial depression in 1920.

9. The average price of gas in dollars per thousand cubic feet for fifty-one cities in the United States is given in the following table:

YEAR	PRICE	YEAR	PRICE	YEAR	PRICE
1913	$0.95	1917	$0.92	1921	$1.32
1914	0.94	1918	0.95	1922	1.28
1915	0.93	1919	1.04	1923	1.26
1916	0.92	1920	1.09		

10. The values, in millions of dollars, of the exports from the United States and from England for a given period were as follows:

YEAR	UNITED STATES	ENGLAND	YEAR	UNITED STATES	ENGLAND
1913	2400	2600	1919	7800	3600
1914	2100	2100	1920	8100	4800
1915	3500	1800	1921	4400	2700
1916	5400	2400	1922	3800	3200
1917	6200	2500	1923	4200	3500
1918	6000	2400			

Plot on the same diagram a solid line for the exports of the United States and a broken line for the exports of England.

QUERIES. Point out the effect of the war on the trade of the United States in relation to the trade of England.

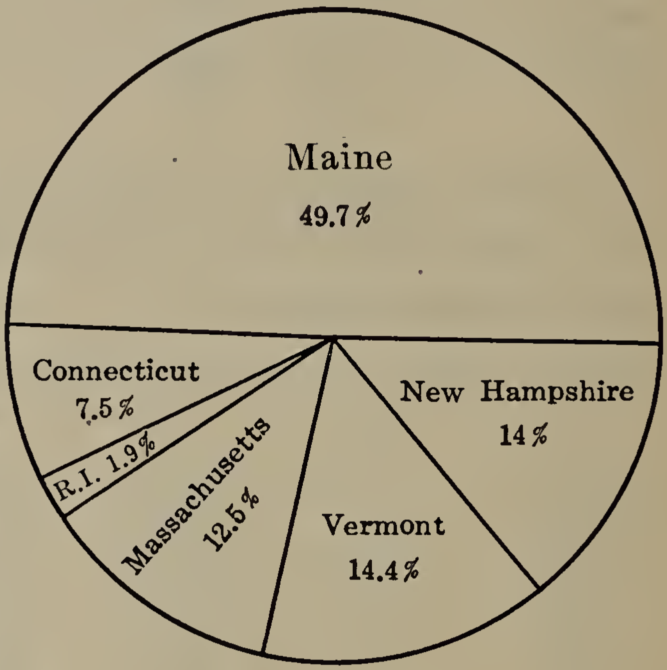
Does the relation seem to be returning to that which prevailed before the war?

19. Circle graphs. When it is desired to show the relation of several quantities to one another and to the whole of which they are parts, the circle graph is useful.

The method of making a circle graph is illustrated on the following page:

EXAMPLE

The areas of the states of Maine, New Hampshire, Vermont, Massachusetts, Rhode Island, and Connecticut are 33,040, 9341, 9564, 8266, 1250, and 4965 square miles respectively. Represent the relation between these areas by a circle graph.



Solution. The sum of the areas of the six states is 66,426 square miles. The percentages of this total area for the various states are Maine 49.7%, Rhode Island 1.9%, Connecticut 7.5%, Massachusetts 12.5%, New Hampshire 14.0%, Vermont 14.4%. Since there are 360° around the center of a circle, the number of degrees representing the area of Maine is 49.7% of 360°, or 178.9°; Rhode Island, 1.9% of 360°, or 7°; Connecticut, 7.5% of 360°, or 27°; Massachusetts, 12.5% of 360°, or 45°; New Hampshire, 14.0% of 360°, or 50°; Vermont, 14.4% of 360°, or 52°.

These figures may be given in tabular form as follows:

STATE	AREA	PER CENT	ANGLE
Maine	33,040 square miles	49.7	179°
New Hampshire .	9,341 square miles	14.0	50°
Vermont	9,564 square miles	14.4	52°
Massachusetts . .	8,266 square miles	12.5	45°
Rhode Island . . .	1,250 square miles	1.9	7°
Connecticut . . .	4,965 square miles	7.5	27°

EXERCISES

1. The area of Europe is about 3,900,000 square miles and is divided as follows:

Cultivated lands	27 %
Prairies	24 %
Forests	28 %
Unproductive lands	21 %

Plot, using circle graph.

2. The area of North and South America is 17,550,000 square miles. Of this area 18 % is mountains, frigid and other unproductive lands; 30 %, forest; 40 %, prairies, pampas, and savannahs; 12 %, cultivated lands. Plot these statistics, using circle graph.

3. The items of expense of a certain family have the relative importance shown below. Indicate this situation by means of the circle graph.

Food	43.1 %
Shelter	17.7 %
Clothing	13.2 %
Fuel, heat, etc.	5.6 %
Sundries	20.4 %
Total	100.0 %

4. The population of the earth, to the nearest 1,000,000 people, is divided, according to races, approximately as follows:

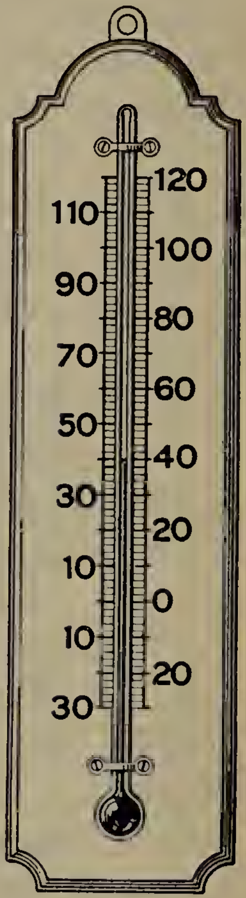
Caucasian (white)	821,000,000
Mongol (yellow)	645,000,000
Semitic	75,000,000
Negro	139,000,000
Other races	68,000,000

Indicate this distribution by means of the circle graph.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

20. Meaning of positive and negative numbers. On the thermometer diagram shown at the right the reading is 40° above zero. If the temperature should drop 25°, the thermometer reading would be 40° − 25°, or 15°. If it should drop 70°, then the reading would be 40° − 70°, or 30° below zero, which is indicated by − 30°, etc. A record such as − 30° means that the reading shows a temperature which is 30° below that reading which is called zero degrees, or 0°. In contrast to the readings below zero, such as − 10°, − 15°, − 20°, etc., the readings above zero are indicated by prefixing plus signs to the numerical records, as in + 10°, + 25°, + 43°, etc.



A similar method is used in recording latitude readings, those in the Northern Hemisphere being plus, or positive, readings, and those in the Southern Hemisphere being minus, or negative, readings. Thus latitude 10° north is expressed + 10° and latitude 10° south is expressed − 10°.

Again amounts of money on deposit in the bank may be regarded as positive and the amounts of overdrafts may be regarded as negative.

21. Illustrations of positive and negative numbers. In general, the uses made of positive and negative numbers arise in connection with measures of quantity which may be regarded as existing in opposite senses; as, for example, money on deposit in a bank and overdrawn accounts, distances measured in opposite directions from a fixed point, time measured before and after a fixed date, etc.

Any rise in the thermometer reading, or upward change, is indicated by placing the sign + before the number of degrees indicating the amount of change; thus, a change of $+15°$ in temperature means a rise of 15°.

Similarly, a fall in the thermometer reading, or downward change, is indicated by placing the sign – before the number of degrees indicating the change; thus, a change of $-10°$ in temperature means a fall of 10°.

From the above it is evident that to indicate a $+15°$ change, that is, to add 15° to a given reading, we count up 15°, and to indicate a $-15°$ change we count down 15°, from the given reading.

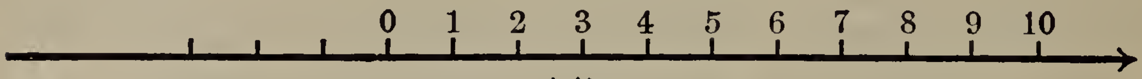
Similarly, the latitude reached by sailing north 20° is indicated by adding $+20°$ to the given latitude reading, and sailing south 20° is indicated by counting down or south 20°, that is, by adding $-20°$ to the given latitude reading.

Likewise, the increasing of one's deposits at the bank is equivalent to adding positive sums to the account, and taking out money from the bank is equivalent to adding withdrawals or negative sums of money.

Thus, $$\$25 + (\$+15) = \$+40,$$

and $$\$100 + (\$-25) = \$+75.$$

22. Addition and subtraction by use of a scale. Let us suppose that equal distances are taken on a line and that the successive points of division are marked with the natural, or positive, numbers as follows:



(*A*)

Such a scale of numbers may be used to illustrate both addition and subtraction as performed in arithmetic.

Thus, in adding 5 to 2 we may begin at 2 and count 5 spaces to the right, obtaining the sum 7. We shall obtain the same result if we begin at 5 and count 2 spaces to the right. This process may be stated in general terms thus:

Rule. *To add the number* a *to the number* b, *begin at* b *and count* a *spaces to the right.*

In subtracting 3 from 5 we may begin at 5 and count 3 spaces to the left, thus obtaining 2. This process may be stated as follows:

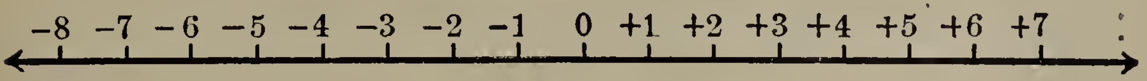
Rule. *To subtract the number* a *from the number* b, *begin at* b *and count* a *spaces to the left.*

If we attempt to subtract 4 from 3 by the preceding rule, we arrive at the first point of division to the left of zero. Arithmetic has no number to represent such a result; in fact, the subtraction of 4 from 3 is there regarded as impossible. We can, however, subtract 3 of the 4 units from the 3 units, leaving one of the 4 units unsubtracted. Now in algebra it is both convenient and necessary to speak of subtracting a greater number from a less, and to call the portion of the greater number, which is unsubtracted, the remainder. The fact that such a subtraction is incomplete is indicated by writing a minus sign before the result; thus, $3 - 4 = -1$. Hence the first point of division to the left of zero may be thought of as corresponding to -1. Similarly, $3 - 5 = -2$; and to -2 may correspond the second point of division to the left of zero.

In like manner $6 - 9 = -3$, which corresponds to the third point to the left of zero. In the same way the fourth point of division to the left of zero would correspond to -4, the fifth point to -5, etc.

Such numbers as -1, -2, -3, etc. are called *negative* numbers. The minus sign is never omitted in writing a negative number, though a letter, as x, may denote one.

The relative order of positive and negative numbers is indicated in the following scale:



(*B*)

ORAL EXERCISES

Perform the following additions and subtractions by counting along the preceding scale:

1. Add 4 to 3.
2. Add 3 to $+4$.
3. Add 7 to -3.
4. Add 4 to -4.
5. Add 2 to -5.
6. Add 6 to -8.
7. Subtract 4 from 6.
8. Subtract 5 from 2.
9. Subtract 5 from 3.
10. Subtract 5 from -3.
11. Subtract 3 from -4.
12. Subtract 4 from -2.

23. Addition of positive and negative numbers. As we have seen, subtraction by the use of scale (B) is performed by counting spaces to the left. Now a negative number represents an unperformed subtraction; therefore to add a negative number to another number means to perform this subtraction.

For example, in subtracting 7 from 4, -3 was obtained by beginning at 4 and counting 7 spaces to the left, arriving at 3 to the left. Hence when we wish to add -7 to any number, we count 7 spaces to the left from that number.

To add -9 to 16 we begin at 16 and count 9 spaces to the left, obtaining 7 as the result; that is,

$$+16+(-9)=+7.$$

Similarly, to add -5 to -3 we begin at -3 and count 5 spaces to the left, obtaining -8 as the result; that is,

$$-3+(-5)=-8.$$

Hence, in general, to add a negative number n to a given number, begin at the given number and count n spaces to the left.

ORAL EXERCISES

Add the following numbers by the use of scale (B):

1. $+5, +2.$
2. $+5, -2.$
3. $-5, +2.$
4. $-5, -2.$
5. $7, -2.$
6. $-7, -1.$
7. $-6, -2.$
8. $6, -3.$
9. $4, -6.$
10. $-3, -4.$
11. $-5, +5.$
12. $-7, +9.$

The preceding exercises illustrate the correctness of the following working rules:

RULE I. *To find the sum of two or more numbers all of which have the same sign, add the numbers arithmetically, and prefix the common sign to the result.*

Rule II. *To find the sum of two numbers with unlike signs, subtract the number with the smaller numeric value from the number with the greater numeric value, and prefix the sign of the greater number to the result.*

These rules are used throughout the whole of algebra. Errors are likely to occur in carrying out processes governed by Rule II. It is important, therefore, that correct habits in adding be formed as rapidly as possible.

The *algebraic sum* of two or more numbers is the number obtained by adding them according to these rules.

The *numeric*, or *absolute*, value of a number is its value regardless of its sign.

Thus, the absolute values of -7, $+10$, and -15 are 7, 10, and 15 respectively. Note that two different numbers, as $+4$ and -4, may have the same numeric, or absolute, value.

The algebraic sum of two numbers is not always the same as the sum of their absolute values; for example, the algebraic sum of $+8$ and -3 is $+5$, but the sum of their absolute values is 11.

Hereafter the word *add* will mean *find the algebraic sum.*

ORAL EXERCISES

Perform the indicated additions:

1. $+6+(+7)$.
2. $-6+(-7)$.
3. $+6+(-7)$.
4. $-6+(+7)$.
5. $+8+(-6)$.
6. $-8+(+6)$.
7. $-7.1+16.3$.
8. $8.3+(-2.6)$.
9. $-5.3+8.7$.
10. $-3+(+3)+(+5)$.
11. $2+(+6)+(+4)+(-3)$.
12. $7+(-2)+(-3)+(+5)$.

Answer the questions asked in the following:

13. $8 + ? = 10$.

14. $8 + ? = 2$.

15. $9 + ? = 12$.

16. $9 + ? = 5$.

17. $-9 + ? = -10$.

18. $-9 + ? = -6$.

19. $-10 + ? = -15$.

20. $-10 + ? = 4$.

21. $12 + ? = 6$.

22. $-12 + ? = 6$.

24. Subtraction of positive and negative numbers. If we wish to subtract 9 from 14, we may do so by answering the question, "What number added to 9 gives 14?" By answering a similar question we can subtract 7 from 13, or 16 from 27, or any number a from another number b. Exercises 13–22, above, are therefore exercises in subtraction, for each asks a question similar to the one in the first sentence of this paragraph.

This point of view brings out the relation that the operation of subtraction bears to that of addition.

ORAL EXERCISES

Perform the following subtractions by answering in each case the question, "What number added to the first number gives the second number?"

Subtract:

1. 4 from 6.

2. 7 from 12.

3. 8 from 4.

4. 12 from 5.

5. -7 from 8.

6. -4 from 4.

7. 6 from -11.

8. -7 from -3.

9. 10 from -17.

10. -25 from 12.

11. In Exercises 1–10, change the sign of the first number (if +, to −; if −, to +) and then add it to the second number. Are the same answers obtained as before?

These results illustrate the following principles:

I. *Subtracting a positive number is the same in effect as adding a negative number of the same absolute value.*

To illustrate: a decrease of $100 in a man's assets is equivalent to an increase of $100 in his liabilities, provided we consider his financial standing as a whole in each case.

II. *Subtracting a negative number is the same in effect as adding a positive number of the same absolute value.*

To illustrate: a decrease of $75 in a man's liabilities is equivalent to an increase of $75 in his assets, as far as his net financial condition is concerned.

Hence, for the subtraction of positive and negative numbers, we have the

Rule. *Change the sign of the subtrahend (if +, to −; if −, to +). Then find the algebraic sum of the subtrahend (with its sign changed) and the minuend.*

ORAL EXERCISES

Subtract the second number (subtrahend) from the first (minuend) in Exercises 1–18:

1. $8, +3$.
2. $8, -3$.
3. $+3, -8$.
4. $+3, +8$.
5. $14, +5$.
6. $14, -5$.
7. $-14, +5$.
8. $-14, -5$.
9. $+7, +7$.
10. $-7, -7$.
11. $7, -7$.
12. $-7, +7$.
13. $-14, -17$.
14. $0, +2$.
15. $-2, -2$.
16. $2, -2$.
17. $-3.5, 2.2$.
18. $6.5, -1.7$.

Supply the missing numbers in the following:

19. $13 - (+2) - (+4) = ?$
20. $-11 - (+2) - (+4) = ?$
21. $-10 - (-2) - (-4) = ?$
22. $18 - (-4) - (+6) = ?$
23. $+7 + ? = 11.$
24. $-7 + ? = -11.$
25. $-4 + ? = 0.$
26. $+5 + ? = 0.$
27. $+5 + ? = 5.$
28. $-7 + ? = -4.$
29. $-8 + ? = -7.$
30. $-8 + ? = 8.$
31. $9 - ? = 4.$
32. $-6 - ? = -4.$
33. $+5 - ? = -8.$
34. $-8 - ? = 5.$
35. $-4 - ? = 0.$
36. $3 - ? = 0.$
37. $4 - ? = 15.$
38. $7 + ? = -3.$

PROBLEMS

1. A thermometer registers − 10°. What is the temperature when the thermometer registers twice as far below zero? three times? four times? five times?

2. One man owes $5. A second man owes twice as much. How much does the second man owe?

3. The car shortage for March on a certain line was 376. This is the same as saying the line had − 376 cars more than were needed. What number would express a shortage twice as great? three times as great?

4. Death Valley is about 325 feet below sea level; that is, its elevation is − 325 feet. The Dead Sea is four times as far below sea level as Death Valley. What number would express the elevation of the Dead Sea?

25. Multiplication of positive and negative numbers. From the preceding problems we see that even in arithmetic we might multiply negative numbers as well as positive numbers. This is more often necessary in algebra than in arithmetic, as algebra deals with negative numbers as well as with positive numbers.

Four cases of the multiplication of numbers may arise; for example,

$$(+3)\cdot(+2)=? \qquad (+3)\cdot(-2)=?$$

$$(-3)\cdot(+2)=? \qquad (-3)\cdot(-2)=?$$

From arithmetic we have learned that $(+3)\cdot(+2)=(+3)+(+3)=+6$.

In $(-3)\cdot(+2)$, -3 is to be added twice, which is the same as adding -6 once; hence $(-3)\cdot(+2)=-6$; in $(+3)\cdot(-2)$, $+3$ is to be subtracted twice, which is the same as subtracting $+6$ once. Therefore $(+3)\cdot(-2)=-6$. Lastly, $(-3)\cdot(-2)$ means that -3 is to be subtracted twice, which is the same as subtracting -6 once. But subtracting -6 is the same as adding $+6$. Therefore

$$(-3)\cdot(-2)=+6.$$

In general terms,

$$(+a)\cdot(+c)=+ac,$$

$$(-a)\cdot(+c)=-ac,$$

$$(+a)\cdot(-c)=-ac,$$

$$(-a)\cdot(-c)=+ac.$$

Therefore we have the

Rule. *The product of two numbers having like signs is a positive number, and the product of two numbers having unlike signs is a negative number.*

ORAL EXERCISES

Find the products of the following:

1. $+3, +4$.
2. $+4, +11$.
3. $-4, +6$.
4. $-2.5, 3$.
5. $-7, +9$.
6. $-7, -5$.
7. $-11, +9$.
8. $+8, -4$.
9. $-7, -8$.
10. $-1.6, -.2$.
11. $+0, +5$.
12. $-8, 0$.
13. $+4, -8, +6$.
14. $+4, -7, -6$.
15. $-4, -6, -2$.
16. $12, +0, -4$.
17. $8, -10, -0$.
18. $-4, +8, -8$.
19. $-3, -4, -5$.
20. $3, -3, +9$.
21. $-1.5, 0.4$.

NOTE. The famous German mathematician Leopold Kronecker (1823–1891) once observed that "the good Lord made the positive integers, but man is responsible for all the rest of the numbers." This expresses the truth about numbers as accurately as one can in a single sentence. We count objects from our earliest years, and so use the positive integers naturally. It is only when we come to study mathematics that the necessity for any other kind of numbers is forced upon us. Here we see that negative numbers are a great convenience if we wish to represent the relations between objects where oppositeness in any of its many forms is involved. But the artificial character of negative numbers delayed their intelligent use for many hundred years. To be sure, the Hindus said that "the square of negative is positive," but the statement probably did not mean anything to those who read it. It was not until after the time of Descartes (see page 284) that the rules for operating on negative numbers were understood, even by great mathematicians.

26. Division of positive and negative numbers. When 16 is divided by 8, the result is 2. To test whether 2 is the correct value of $16 \div 8$, we check thus: $8 \cdot 2 = 16$. To check $12 \div 4 = 3$, we multiply the quotient 3 by the divisor 4 and get 12, the number divided. This test will be used to determine whether when positive and negative numbers are divided, the answer is positive or negative. All the cases which may arise are represented by the four questions:

$(a) +16 \div (+8) = ?$ $\qquad$ $(c) +16 \div (-8) = ?$

$(b) -16 \div (+8) = ?$ $\qquad$ $(d) -16 \div (-8) = ?$

These questions are answered thus:

$(a) +16 \div (+8) = +2$, because $+2 \cdot (+8) = +16.$

$(b) -16 \div (+8) = -2$, because $-2 \cdot (+8) = -16.$

$(c) +16 \div (-8) = -2$, because $-2 \cdot (-8) = +16.$

$(d) -16 \div (-8) = +2$, because $+2 \cdot (-8) = -16.$

In (*a*) and (*d*) the dividends and the divisors have like signs and the quotients are positive. In (*b*) and (*c*) the dividends and the divisors have unlike signs, and the quotients are negative.

Therefore we have the

Rule. ***When two numbers having like signs are divided, the quotient is a positive number; when two numbers having unlike signs are divided, the quotient is a negative number.***

27. Division by zero. The result of multiplication by zero is given a definite meaning in arithmetic and algebra, namely zero; but in both subjects division by zero is always excluded. If zero were used as a divisor, numerous contradictions would arise, of which the following is an illustration:

Obviously, $\qquad 0 \cdot 3 = 0,$

and $\qquad 0 \cdot 5 = 0.$

Therefore $\qquad \cancel{0} \cdot 3 = \cancel{0} \cdot 5.$

Dividing each by zero, $\qquad 3 = 5,$

which is false.

NOTE. The Hindus were the first to express the laws that govern the operations with the number 0. In fact, they were the first to have such a symbol. In the twelfth century a Hindu writer stated that $a + 0 = a$, that $\sqrt{0} = 0$, and that $0^2 = 0$. Of course he did not express himself in terms of these symbols, but in the notation of his time and country.

ORAL EXERCISES

In Exercises 1–12 divide the first number by the second:

1. $+10, +5$.
2. $-10, -2$.
3. $-15, +5$.
4. $+14, -7$.
5. $-21, -3$.
6. $-6, +6$.
7. $0, +4$.
8. $-4, -4$.
9. $-12, +3$.
10. $+12, -3$.
11. $-12, -3$.
12. $+36, 2$.

MISCELLANEOUS ORAL EXERCISES

Simplify the following:

1. $(6) + (5)$.
2. $(6) - (5)$.
3. $(6) - (-5)$.
4. $(-6) + (5)$.
5. $(-6) - (-5)$.
6. $-6 + 5$.
7. $(-8) - (4)$.
8. $-8 - 4$.
9. $-11 + (-12)$.
10. $-6 - (-9)$.
11. $-6 + 9$.
12. $9 + (-11)$.
13. $12 - 16$.
14. $-16 - 12$.
15. $18 - 16$.
16. $+8 - 0$.
17. $0 - 2$.
18. $(-3)(5)$.
19. $(-3)(7)$.
20. $(-7)5$.
21. $8(-4)$.
22. $(-5)(-11)$.
23. $(-3)8$.
24. $-4 \cdot 6$.
25. $6 \cdot 0$.
26. $0 \cdot (-8)$.
27. $5 \cdot 8$.
28. $-3 \cdot 7$.
29. $12 \div (-3)$.
30. $-6.8 \div (-0.2)$.
31. $-38 \div (-2)$.
32. $45 \div (+15)$.
33. $-45 \div (+15)$.
34. $0 \div (-7)$.
35. $0 \div 4 \div (-5)$.
36. $8.4 \div .4$.
37. $3 - 7 + 8$.
38. $-4 + 5 - 2 + 1$.

Supply the missing term in the following:

39. $\frac{-20}{?} = 5.$ **40.** $\frac{-12}{?} = 2.$ **41.** $\frac{+36}{?} = -4.$

Add:

42. $\begin{array}{r} 8 \\ -2 \\ 3 \\ -6 \\ \hline \end{array}$ **43.** $\begin{array}{r} 6 \\ -2 \\ +3 \\ -4 \\ \hline \end{array}$ **44.** $\begin{array}{r} 8 \\ -6 \\ -2 \\ 5 \\ \hline \end{array}$ **45.** $\begin{array}{r} -4 \\ +9 \\ +3 \\ -6 \\ \hline \end{array}$ **46.** $\begin{array}{r} 8 \\ -7 \\ -5 \\ +4 \\ \hline \end{array}$

Simplify:

47. $3 \cdot 8 \div 2.$

48. $-3(8) \div (-2).$

49. $3(-8) \div 2.$

50. $4 \cdot 6(-7) \div (-16).$

51. $18 \div (3) \cdot 6 \div (-4).$

52. $2^2.$

53. $(-2)^2.$

54. $(3)^3.$

55. $(-3)^3.$

56. $(-3)^3 + (3)^3.$

57. $-(-4)^2.$

58. $-(-4)^3.$

59. $-2(3)^2(2).$

60. $-3(-2)(-3)^3.$

61. $-2(-4)^2(-5).$

62. $2(4)^2 \cdot 5.$

63. $3(5)^2 \cdot (-6).$

64. $-4(5)^2 \div 2.$

65. $-4(5)^2 \div (-2).$

66. $-4(6)^2 \div 6.$

67. $4(-6)^2 \div (-6).$

68. $3 \cdot 5(-7)^2 \div 21.$

69. $3 \cdot 5(7)^2 \div (-21).$

70. $-4 \cdot 2 \cdot (9)^2 \div (6)^2.$

REVIEW EXERCISES

Simplify the following:

1. $4^2 - (2)^2.$

2. $4^2 + (-2)^2.$

3. $3^3 - (-2)^2.$

4. $3^3 + (-2)^3.$

5. $(5-2)(3+2).$

6. $(5-3)(4+2).$

7. $(5-2)(7-3) \div (3-9).$

8. $(-1)^2 + (-1)^3 - (-2)^2 - (-2)^3.$

9. $9 + 3 \cdot 2 - 18 \div 3$.

10. $5^2 + 4 \div 2 \div (-6)(-3)$.

11. $3 \cdot 6 \div 9 + 2 \cdot 6 \div 4 - (-3)^2$.

12. $(-3)^2 + 2 \cdot 3 \cdot (-4) \div 4^2$.

13. $(+2)^2 - 2(2)(-3)^2 + (-3)^3$.

14. $3(4)^2(-5) - (-5)^2$.

15. $4^3 - 3(4)^2(-3) + 3(4)(-3)^2 - (-3)^3$.

16. $3^3 + 3(3)^2(-2)^2 + 3(3)(-2)^3 + (-2)^4$.

17. $(-3)^3 - 3(-3)^2(0) + 3(-3)(0)^2 - 0^3$.

18. $(-5)^2 + 3(-5)^2(+4) + 3(-5)(+4)^2 - (4)^3$.

19. $2(-3)^3 + 6(-5)^2(+4) + 6(-5)(+4)^2 + (4)^3$.

If $x = -3$, $y = 2$, find the value of the following:

20. y^2.
21. y^3.
22. x^2.
23. x^3.
24. y^4.
25. x^4.
26. $3y^2$.
27. $3x^2$.
28. $3y^3$.
29. $3x^3$.
30. $5xy$.
31. $5x^2y^2$.
32. $x^2 + y^2$.
33. $x^3 + y^3$.
34. $2x^2 + 4xy + y^2$.
35. $2x^2 - 4xy + y^2$.
36. $x^3 - 3x^2y + 3xy^2 - y^3$.
37. $x^4 - y^4$.
38. $3x^2 + 6xy + 3y^2$.
39. $3x^2 - 6xy + 5y^2$.
40. $(x + y)^2(x - y)^2$.
41. $2x^3 + 6x^2y + 6xy^2 + 2y^3$.

42. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

43. Does $4x - 2 = 2x - 12$, if $x = -5$?

44. Does $3x - 5 = 2x + 8$, if $x = 12$?

45. Does $x^2 - x - 12 = 0$, if $x = 3$? if $x = -3$? if $x = +4$?

46. Does $x^2 + 10x = -25$, if $x = \frac{1}{2}$? if $x = -4$? if $x = -5$?

47. Does $2x^2 - 20x + 50 = 0$, if $x = -5$? if $x = 5$? if $x = \frac{1}{3}$?

48. Does $x^2 + x - 12 = 0$, if $x = +4$? if $x = -8$? if $x = -4$? if $x = 3$?

49. Does $x^2 - 7x = -12$, if $x = 6$? if $x = \frac{1}{2}$? if $x = -3$?

50. Using positive numbers to represent excess, and negative numbers to represent shortage, express the following records on the number of freight cars available during the first six months of a certain year: January, excess of 560 cars; February, excess of 420 cars; March, shortage of 176 cars; April, shortage of 360 cars; May, shortage of 574 cars; June, excess of 564 cars.

51. The temperature is + 15°. What will represent the temperature after a drop of (*a*) 5°? (*b*) 10°? (*c*) 20°?

52. The temperature is now − 18°. What will it be (*a*) after a rise of 8°? (*b*) after a rise of 14°? (*c*) after a rise of 30°? (*d*) after a change of − 10°? (*e*) after a change of + 30°?

53. The temperature at 6 A.M. was − 12°. During the day it rose at the rate of 3° per hour. What was the temperature at 9 A.M.? at 10 A.M.? at 12 M.?

54. The speed of a trolley car going in a certain direction, taken at half-hour intervals, was as follows: 20 mph (miles per hour), 25 mph, 15 mph, 0, 10 mph, 0, − 25 mph, − 10 mph, 0, − 5 mph. Plot these values, using the line graph. What is the meaning of the negative velocities given?

Where an initial direction has been assumed, we are frequently confronted with negative directions. This is particularly true in physics and the applied sciences.

55. A man withdraws $50 from his bank one week. The second week he withdraws $10 less than the first week; the third week $10 less than the second week; and so on for a period of eight weeks. Represent these withdrawals by means of a line graph. Interpret the negative values found after the sixth week.

CHAPTER IV

ADDITION

28. Monomials. A *monomial,* or *term,* is a number symbol which is not the indicated sum or difference of two or more number symbols.

Thus 6, $-b$, a^2, b^2c, and $-3\,cd^2$ are monomials, or terms.

Frequently, where no confusion would arise, expressions like $(5-3)$, $2(x+y)$, $4\sqrt{x^3}$, and $\sqrt{b-x}$ are called terms, for in such cases one thinks not of the parts but of a single number for which the whole stands. We may think of the expression $(5-3)$ not as two numbers but as the one number 2.

29. Similar terms. Terms are *similar* when they are alike in all respects except their coefficients.

Thus 2, -6, and 8 are similar terms, as are b, $3\,b$, and $-5\,b$. Also $\sqrt{5}$ and $2\sqrt{5}$ are similar terms, as are $2\,a^2x$, $-5\,a^2x$, and $6\,a^2x$.

30. Addition of similar terms. It is clear that $6\,b+4\,b=10\,b$; also, that $3\,ax+2\,ax+8\,ax=13\,ax$. In like manner $4\,abc+(-3\,abc)+12\,abc+(-8\,abc)=5\,abc$, and the sum of $3\,xy$, $-xy$, $8\,xy$, $-4\,xy$, and xy is $7\,xy$. The terms $-xy$ and $+xy$ are equivalent to $-1\,xy$ and $+1\,xy$.

For adding similar terms we have the

RULE. *Find the algebraic sum of the numeric coefficients and prefix this result to the common literal part.*

ORAL EXERCISES

Find the algebraic sum of:

1. $5b$, $3b$.
2. $-12a$, $+7a$.
3. $-9c$, $-3c$.
4. $-2a$, $-3a$.
5. $+8x$, $-2x$.
6. $+5r$, $-8r$.
7. $-11a$, $+19a$.
8. $-2ac$, $-3ac$.
9. $-xy$, $+3xy$, $-4xy$.
10. $-6mn$, $+2mn$, $-mn$.

Combine:

11. $ax + 2ax - 8ax + 4ax$.
12. $3am - 5am - 9am + 7am$.
13. $5ab - 4ab + 11ab - 8ab$.
14. $9ab - 7ab + 4ab - 5ab$.
15. $4axy + 3axy - 7axy + 14axy$.
16. $-abc + 4abc + abc - 5abc + 11abc$.
17. $10bc - 6bc + 12bc - 8bc$.
18. $4bc - 7bc + 10bc - 8bc + 12bc$.
19. $6xy$, $-xy$, $-7xy$, $+xy$, $8xy$, $-4xy$, xy.
20. $3(x+y)$, $-4(x+y)$, $+6(x+y)$, $-(x+y)$, $-5(x+y)$.

31. Order in adding terms. Obviously, $3+4+6=4+6+3=6+4+3$, etc. This illustrates the law that in addition the terms may be arranged and added in any order.

Hence $5b+6c=6c+5b$, and the sum of 4 and b is either $4+b$ or $b+4$; also $b+c=c+b$ and $x+y+z=z+y+x$.

32. Addition of dissimilar terms. An algebraic expression for the sum of two terms which are not similar, such as $5x$ and $3k$, is obtained by writing them one after another with the plus sign between them; thus, $5x+3k$.

Similarly, the sum of $4a$ and $-5m$ is $4a+(-5m)$ or $4a-5m$, and the sum of $2x$, $4y$, and $-2z$ is $2x+(+4y)+(-2z)$, or, omitting the parentheses, $2x+4y-2z$.

For adding dissimilar terms we have then the

Rule. *Write the terms one after the other in any order, giving to each its proper sign.*

EXERCISES

Write the sum of:

1. $2a$, $3b$, $-2c$.
2. $3x$, $-b$, $5y$, 8.
3. $2a^2b$, $3bx$, $-2cy$, ab^2.
4. $6x^3y$, $-4xy^3$, $2c^3y$, $-cy^3$.
5. $5x$, $-2a$, $3b$, $-4x$, $4y$.
6. $2a^2, +3b^2, -8c^2, -5b^2, 6a^2$.
7. $4a$, $-7b$, $+4c$, $6b$, $-3c^2$.
8. $4a^3b$, $-4ab^3$, $-4a^2b$, $4ab^2$, $3a^3b$, $3ab^3$.
9. $7a^2bc$, $-5abc^2$, $9ab^2c$, $7abc^2$, $-3a^2bc$, $-6ab^2c$.
10. $10a^3b$, $3ab^3$, $-4a^2b$, $-7ab^3$, $6a^2b^3$, $-3a^2b$.

Simplify:

11. $5-9-2+15-20+10$.
12. $14ac-8xy+4ac-3xy-7ac+12xy$.
13. $8x+7a-14a-9x+4x$.
14. $12y-18y+9b+25y-24b$.
15. $4z^2-5y^2-10y^2+z^2-8y^2$.
16. $5ab-7xy-11ab+14ab-xy-7ab$.
17. $-4ab^2+6xy^2-13ab^2+4xy^2+ab^2-0xy^2$.
18. $7a^3+4a^2b-6ab^2+a^3+5b^3+5a^2b-2b^3+3ab^2$.
19. $7b^2d-5ab^2+6b^2d-ab^2+9ab^2-10b^2d$.
20. $-b^4-18a^2+19b^4-0b^4+13a^2+b^4-0a^2$.
21. $10ab^3+4cd^2-0ab^3+14ab^3-11ab^3-23cd^2+0cd^2$.

22. $11\sqrt{b} - 15\sqrt{b} + 20\sqrt{b} - 0\sqrt{b}$.

23. $4\sqrt{a-b} - 2\sqrt{a-b} + 7\sqrt{a-b} - 6\sqrt{a-b}$.

24. $5(2a-b) + 2(2a-b) - 4(2a-b) - (2a-b)$.

25. $4\sqrt{a^2b} + 3\sqrt{a^2b} - 7\sqrt{a^2b} + 4\sqrt{a^2b} - 9\sqrt{a^2b}$.

33. Addition of polynomials. A *polynomial* is an algebraic expression consisting of two or more terms.

An expression is not called a polynomial if any of its terms contain a letter under the radical sign. Thus $\sqrt{a} - 2 + 3$ is not called a polynomial.

A *binomial* is a polynomial of two terms.

Thus $x + 3$, $3x - 5$, $h^2 - 9$, and $3 - 2x^2$ are binomials.

A *trinomial* is a polynomial of three terms.

Thus $x - y - 3$, $2x + y - 8$, $3x + 4y - 7z$, $m - n - p$, and $2x + 6r - 5s$ are trinomials.

EXAMPLE

Add the following polynomials: $2a - 3b - ac^2$, $2b + 3ac^2$, $-3a - 7ac^2 + 11$, and $4a + 3b - 5$.

Solution.

$$\begin{array}{lrrrr}
 & +2a & -3b & -ac^2 & \\
 & & +2b & +3ac^2 & \\
 & -3a & & -7ac^2 & +11 \\
 & +4a & +3b & & -5 \\
\hline
\textit{Sum,} & 3a & +2b & -5ac^2 & +6
\end{array}$$

For the addition of polynomials we have the

RULE. *Write similar terms in the same column.*

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

34. Checks. A *check* on an operation is a second operation which tests the correctness of the first.

For example, in arithmetic the result of subtraction is checked by addition; thus, the check for $10 - 4 = 6$ is $6 + 4 = 10$. The check for the result of division is multiplication. Thus the check for $144 \div 8 = 18$ is $18 \cdot 8 = 144$.

To check addition we sometimes add columns in the opposite direction. However, there is really no check for addition that is absolutely certain to detect errors. If the numbers in the column are not all of the same sign, the result can be partially checked by adding the positive and negative numbers separately and finding the algebraic sum of the two results.

Perhaps the most common check for the addition of polynomials is to substitute some number or numbers for the letters in the expression. Each term which is being added will then have some numeric value. The numeric value of the algebraic sum of the polynomials should be equal to the algebraic sum of the numeric values of the individual polynomials. If this does not turn out to be the case, some error in the original addition has been made.

Thus, in the example given above, let $a = 1$, $b = 2$, and $c = 3$. The work of addition and the corresponding numeric check may then be represented as follows:

$$
\begin{array}{rrrrlrrrrl}
2a & -3b & -ac^2 & & (= & 2 & -6 & -9 & & =-13) \\
 & +2b & +3ac^2 & & (= & & +4 & +27 & & =+31) \\
-3a & & -7ac^2 & +11 & (= & -3 & & -63 & +11 & =-55) \\
4a & +3b & & -5 & (= & 4 & +6 & & -5 & =+\ 5) \\
\cline{1-4}\cline{6-10}
3a & +2b & -5ac^2 & +\ 6 & (= & 3 & +4 & -45 & +\ 6 & =-32)
\end{array}
$$

The sum of the numeric values of the several expressions is in this case the same as the numeric value of the sum of the expressions, namely -32, thus indicating, although not proving, that the addition has been correctly done.

EXERCISES

Add the following polynomials and check the results:

1. $$\begin{array}{r} x+2\\ 3x-4\\ \underline{4x-6} \end{array}$$

2. $$\begin{array}{rrr} x- & z & -2\\ 2x+ & z & -7\\ \underline{6x-} & \underline{3z} & \underline{+9} \end{array}$$

3. $$\begin{array}{rrr} x- & y+ & z\\ 2x- & 3y+ & 3z\\ \underline{4x+} & \underline{3y-} & \underline{6z} \end{array}$$

4. $4x+4y$, $5x-9y+4z$, and $4x-4y-4z$.

5. $8x-7y+4z$, $6x-5z$, and $3x+7y-6z$.

6. x^2-x-2, $2x^2-4x+6$, and $4x^2+7x-9$.

7. x^2-5x+2, $2x-4x^2+8$, and $11x^2-10x-18$.

8. $3d-4d^2-6$, d^2+d-1, $4d-3d^2+3$, and $4-d-5d^2$.

9. $4a-5b+6c$, $7b-2c$, and $6a-4b-3c$.

10. $a+2b+3c$, $b-3c+a$, and $c-2a-4b$.

11. $6x-7y+8z$, $2y-12z+x$, and $10z-4y$.

12. $4a-3b-3c$, $3c-2a-b$, and $4b+8c$.

13. $10bc-ac$, $7ab-3bc$, and $-12bc-ab$.

14. $2a^2-4a+5$, $5a-3a^2+2$, $6a-4a^2$.

15. $3-a^2+3a$, $-6+3a^2-2a$, $6a^2-4a$.

16. $a-x-4c$, $5-6x+5a$, $6c-3a+9$, and $a-2c-2x-12$.

17. $a+c+x-5$, $3a+2x+9-2c$, $a-7-2c-x$, and $6-3x-4a+c$.

18. $x-2-4c$, $x-a-3c$, $a-4c-6$, and $3x-7+12a$.

19. $3x+4y+6z$, $2x-2y+4z$, and $4x-3y+11z$.

20. $2x-10z$, $4x+3y$, $5z-y$, and $3x-4y+z$.

21. $2-4(x-y)+2(x+z)$, $5-10a+3(x-y)$, $-2(x+z)+7$, and $5(x-y)+(x+z)+3a$.

22. $a+c-2(b-d), 7(b-d)+6c-11a$, and $10a-4(b-d)$.

23. $a+x+3(y+z)$, $2a-3x+4b$, $7x-10(y+z)$, and $7b-3a+z+2(y+z)$.

Combine similar terms in the following polynomials:

24. $6a^2-12b^2+11c^2+14b^2-2a^2-c^2+3b^2+4c^2$.

25. $a^2+2ab+b^2-3ab-4a^2-b^2+8a^2-4ab+b^2$.

26. $5bc-b^2+c^2+4bc-3c^2+5b^2-4bc-5c^2$.

27. $5w^2-6w+11-4w-8-3w^2+7w-18+13w$.

28. $4y^2z-yz+z^2-3yz+4yz^2-2y^2z+4z^2+3yz^2+y^2z$.

29. $\frac{1}{2}b+\frac{1}{3}c-\frac{1}{4}d+5$, $\frac{1}{2}c-\frac{2}{3}b-c$, $\frac{1}{2}d+\frac{3}{4}b+7$.

The sum of $2a$ and $3a$ may be written $(2+3)a$. This is not usually done, as the sum of $2a$ and $3a$ can be written $5a$. In adding ax and $3x$ the a and 3 cannot be combined. The sum $ax+3x$ can be written $(a+3)x$. Similarly, $bx-5x=(b-5)x$; $bx+x=(b+1)x$; and $2x+ax+cx=(2+a+c)x$; also $a(y+z)+c(y+z)=(a+c)(y+z)$.

Express as one term so that x will have a binomial or a polynomial coefficient:

30. $ax+4x$.

31. $2ax+bx$.

32. $4bx-x$.

33. $ax+2bx+cx$.

34. $ax-3x+bx$.

35. $4ax-4cx-x$.

36. $2ax-2bx-x+b^2x$.

37. $cx-5bx-x+3cx$.

38. $2x-3ax+bx-cx$.

39. $.3x+2.5ax-.7bx+.5cx$.

CHAPTER V

SIMPLE EQUATIONS

35. Definitions. An equation has already been defined as a statement of equality between two equal numbers or number symbols.

Thus $3 = 7 - 4$, $x - 2y = 3x + y - 2x - 3y$, $4a = 2 + 12$, and $x^2 - x - 6 = 0$ are equations.

The part of the equation on the left of the equality sign is called the *first* or *left member*; that on the right, the *second* or *right member*.

In an equation the letter whose value is sought is called the *unknown letter* or the *unknown.*

Thus, in the equation $2x = 4$, x is the unknown, and in the equation $3h - 5 = 13$, h is the unknown.

The process of finding the value of the unknown letter in an equation is called *solving the equation.* If the value of the unknown be substituted for it in the equation and if, when the result is simplified, the left member of the equation becomes identical with the right, then the unknown is said to *satisfy* the equation.

The presence of an equality sign between two algebraic expressions is not sufficient to form an equation.

Thus $x = x + 1$ is not an equation, since there is no number which equals itself increased by 1.

ORAL EXERCISES

Does the number in parenthesis at the right of each equation satisfy that equation?

1. $x - 3 = 2.$ (5)
2. $4 - y = 2.$ (3)
3. $3 + t = 6.$ (3)
4. $4 - m = 2.$ (2)
5. $2z + 3 = 5.$ (1)
6. $5 - 3x = 5.$ (1)
7. $2y + 5 = 11.$ (3)
8. $6y - 10 = 2.$ (1)
9. $4 - 5y = -1.$ (6)
10. $2m - 2 = 0.$ (1)
11. $4x - 5 = 7.$ (3)
12. $12s + 5 = 29.$ (1)

36. Axioms. An *axiom* is a statement whose truth is accepted without proof.

In solving equations, constant use is made of the following axioms:

Axiom I. *If the same number is added to each member of an equation, the result is an equation.*

Thus, adding 4 to each member of $n - 4 = 7$ gives $n = 11$. In this case the use of Axiom I gives us the solution to the original equation.

ORAL EXERCISES

Find the value of the unknown in:

1. $x - 10 = 7.$
2. $n - 3 = 5.$
3. $d - 4 = 4.$
4. $x - 3 = 6.$
5. $n - 2 = 13.$
6. $s - 1 = 3.$
7. $-10 + r = 3.$
8. $-4 + m = 1.$
9. $n - 2 = 0.$
10. $x - 8 = 0.$
11. $-6 + c = 7.$
12. $-5 + d = 3.$
13. $0 = x - 2.$
14. $h - 5 = 0.$
15. $0 = -3 + x.$

Axiom II. *If the same number is subtracted from each member of an equation, the result is an equation.*

Thus, subtracting 5 from each member of $n + 5 = 10$ gives $n = 5$. Axiom II states that if this subtraction is performed, the resulting statement is true.

ORAL EXERCISES

Find the value of the unknown in:

1. $x + 5 = 8$.
2. $a + 3 = 7$.
3. $r + 15 = 21$.
4. $5 + d = 20$.
5. $2 + m = 20$.
6. $h + 5 = 7$.
7. $10 + b = 16$.
8. $3 = x + 2$.
9. $9 = a + 4$.
10. $12 = d + 5$.
11. $12 + x = 15$.
12. $x + 2 = 10$.
13. $2 + y = -3$.
14. $m + 10 = 16$.
15. $y + 3 = 16$.
16. $3 + x = 5$.
17. $2 + t = -8$.
18. $3 + v = 9$.
19. $12 + x = 2$.
20. $x + 4 = 1$.
21. $z + 11 = 20$.

Axiom III. *If each member of an equation is multiplied by the same number, the result is an equation.*

When an equation is in the form $\frac{n}{3} = 5$, it can be solved by multiplying each member by 3, giving $n = 15$. Axiom III states that if this multiplication is performed, the result is true.

ORAL EXERCISES

Find the value of the unknown in:

1. $\frac{x}{3} = 1$.
2. $\frac{x}{2} = 4$.
3. $\frac{a}{5} = 8$.
4. $\frac{x}{2} = 6$.
5. $\frac{n}{4} = 10$.
6. $\frac{n}{4} = \frac{1}{2}$.
7. $\frac{k}{6} = \frac{1}{3}$.
8. $\frac{r}{4} = -3$.
9. $\frac{1}{4} = \frac{l}{8}$.
10. $\frac{1}{5} = \frac{m}{10}$.

Axiom IV. *If each member of an equation is divided by the same number (not zero), the result is an equation.*

In the equation $4n = 12$, divide both members by the coefficient of n. Thus, dividing both members by 4 we get $n = 3$. Axiom IV states that if this division is performed, the result is an equation.

ORAL EXERCISES

Find the value of the unknown in:

1. $6a = 12$.
2. $8n = 24$.
3. $7d = 28$.
4. $5n = 30$.
5. $3k = 15$.
6. $2m = 6$.
7. $12x = 60$.
8. $24 = 8m$.
9. $56 = 14n$.
10. $18 = 6r$.
11. $125 = 25x$.
12. $13y = 65$.

State the axiom or axioms which apply to the solution of each of the following equations:

13. $3n - 7 = 2$.
14. $2x + 2 = 4$.
15. $5r - 3 = 12$.
16. $7s - 2 = 12$.
17. $\frac{3c}{2} = 6$.
18. $8h = 16$.
19. $7d + 7 = 21$.
20. $3x - 4 = 5$.
21. $8h + 2 = 10$.
22. $\frac{2r}{5} = 4$.
23. $6l = 18$.
24. $\frac{4x}{3} = 4$.

In changing the form of the equation, by the application of an axiom, we do not change the value of the unknown. We merely discover the value it had all the time. These axioms are used almost constantly in the solution of all kinds of equations, whether they involve numbers, letters standing for known numbers, unknown numbers, or all of these together.

Example

Solve $6x - 7 = 3x + 2$.

Solution. $6x - 7 = 3x + 2$.

Subtracting $3x$ from each member,

$$3x - 7 = 2. \qquad \text{Ax. II}$$

Adding 7 to each member,

$$3x = 9. \qquad \text{Ax. I}$$

Dividing each member by 3,

$$x = 3. \qquad \text{Ax. IV}$$

Checking the solution of an equation is often called *testing* or *verifying the result.*

Check. $6x - 7 = 3x + 2$.

Substituting 3 for x,

$$6 \times 3 - 7 = 3 \times 3 + 2.$$

Simplifying, $18 - 7 = 9 + 2$, or $11 = 11$.

Hence x satisfies the original equation.

EXERCISES

Solve the following equations and verify the results:

1. $x + 4 = 7$.
2. $y - 8 = 5$.
3. $3x - 4 = 14$.
4. $5x - 2 = 3x + 6$.
5. $5t = 3t + 8$.
6. $4t = t + 3$.
7. $3x + 2 = x + 6$.
8. $4t + 2 = 3t - 4$.
9. $4t + 2 = 2t$.
10. $4t + 2 = t + 5$.
11. $4m - 7 = 2m + 3$.
12. $3l - 5 = 7 - l$.
13. $6 - b = 9 - 4b$.
14. $2x = 6 - x$.
15. $5 + 3t = t + 3$.
16. $5t - 1 = 2t + 8$.
17. $5x - 1 = 2x + 5$.
18. $n + 14 = 2 - 3n$.
19. $4w - 2 = 7 + w$.
20. $4t + 3 - t + 5 = t - 10$.

21. $3 + 2z - 1 = 12 - 3z$.

22. $4m - 3 + 3m - 4 = 6m - 8$.

23. $36 + 8m + 4 - 9m = 2m + 10$.

24. $3m + 12m + 17 - m + 4m = 107$.

25. $4h - 3 = 5h - 16 - 43h - 71$.

26. $6x - 3 - 7x - 2 = 4x + 7 - 3x - 4$.

27. $4y + 6 - 3y + 8 = 2y + 7 - 3y - 8 + 5y$.

28. $3x+7-4x+2+5x=2x+4-5x+7-3x+22+6x$.

ORAL EXERCISES

1. The side of a rectangle is x feet. How long is the side of a rectangle 3 feet shorter? of one 4 feet shorter? of one 5 feet shorter?

2. One rectangle is n feet long. How long is a rectangle 3 feet longer? 4 feet longer? 5 feet longer?

3. The side of a square is x feet. What is the side of a square 2 feet shorter?

4. A rectangle contains x square feet. How many square feet does it contain after it is diminished by 30 square feet? by 50 square feet? by 100 square feet?

5. A park contained x acres. How large was it when it was increased by 200 acres?

6. The area of a triangle is m square feet. What is the area of a triangle twice as large? 3 times as large? 4 times as large?

7. A triangle contains 10 square feet. What is the area of a triangle x times as large?

8. A bin contained t tons of coal. Four tons were removed. How much coal remained?

9. A bin contained 15 tons of coal after x tons were removed. How many tons did it contain at first?

10. The area of a circular pond is k square yards. What is the area of a pond that contains 200 square yards more? 250 square yards more? x square yards more?

11. The area of a circular pond containing x square feet is increased by 30 square feet. How many square feet does it now contain?

12. At the end of a dry summer the area of a certain lake whose area was originally x acres was found to have diminished 300 acres. What was its new area?

13. The volume of a pyramid is x cubic yards. What is the volume of a pyramid 3 times as large? 5 times as large?

Express the following statements as equations:

14. The sum of 3 and x is 10.

15. The sum of x and 2 is 12.

16. x is equal to 5 increased by 2.

17. x diminished by 5 equals 7.

18. x is equal to 10 diminished by 6.

19. x increased by 5 equals 3.

20. Three times x increased by 4 equals 16.

21. Three times a is 12 more than 15.

22. Four times x diminished by 8 equals two times x.

23. One half of a diminished by 5 equals 7.

24. One third of a increased by 10 equals four times a decreased by 1.

25. $2x$ increased by 5 is the same as $3x$ divided by 4.

26. Four times x diminished by 7 equals three times x added to 3.

27. One half of x increased by 5 equals two times x diminished by 10.

EXAMPLE

In his will a man bequeaths $7000 to his wife and daughter with the provision that the wife is to receive $1000 more than twice the amount received by the daughter. How much does each receive from his estate?

Solution. By the conditions of the problem the wife and daughter together receive $7000 and the wife receives $1000 more than twice that received by the daughter. The problem requires that the amount received by each be determined.

(In order fully to grasp the conditions of a problem, it is frequently necessary to make such a restatement as the above, although usually it need not be written down.)

The implied equation is:

Amount received by the wife + amount received by the daughter = $7000.

Let $d =$ number of dollars received by the daughter.

Then $2d + 1000 =$ number of dollars received by the wife.

Hence
$$2d + 1000 + d = 7000,$$
$$3d = 6000,$$
$d = 2000$, the number of dollars received by the daughter.

$2 \times 2000 + 1000 = 5000$, the number of dollars received by the wife.

Check.
$$\$5000 + \$2000 = \$7000.$$
$$\$5000 = 2 \times \$2000 + \$1000.$$

The preceding solution leads to the following

Rule. *In the solution of problems involving the use of simple equations the following steps are necessary:*

1. Read the problem carefully and find the statement which will later be expressed in an equation.

2. State briefly the implied equation in words, using the symbols of operation +, −, ×, ÷, and the equality sign.

3. Represent the unknown numbers by means of numerals and letters.

4. Express the conditions stated in the problem as an equation involving these symbols.

5. Solve the equation.

6. Check, by substituting in the problem the values found for the unknowns.

PROBLEMS

1. One boy has 10 more marbles than another. Together they have 52. How many has each?

2. Two boys caught 27 fish. One caught 5 more than the other. How many did each catch?

3. Two girls sell 37 bags of pop corn at a bazaar. One sells 3 more than the other. How many does each sell?

4. The perimeter of a rectangle is 46 feet. It is 3 feet longer than it is wide. What are its dimensions?

5. The perimeter of a rectangle is 50 feet. It is 4 times as long as it is wide. What are its dimensions?

6. One number is 4 times another number. Their difference is 36. Find the numbers.

7. A rectangle is 10 feet longer than it is wide. Its perimeter is 70 feet. What are its dimensions?

8. One circle is half as large as another. Their combined areas are 30 square feet. What is the area of each?

9. The perimeter of a triangle is 41 yards. The first side is 4 yards shorter than the second, and the third is 3 yards less than the second. What is the length of each side?

10. A flagpole is half as tall as the building on which it stands. The flag at full mast is 150 feet above the ground. How long is the pole?

11. The sum of three numbers is 77. The second is twice the first and the third 5 more than three times the first. What are the numbers?

12. One fish pool is twice as large as another. In these two pools 72 fish are to be placed in numbers proportional to the size of the pools. How many fish should be placed in each pool?

13. A poultryman has three yards. The first is three times as large as the second, and the second is twice as large as the third. He wishes to separate his flock of 360 chickens into flocks proportionate to the size of the yards. How many chickens should he place in each yard?

14. A farmer's flock of 300 sheep is to be placed in three fields so that the second shall contain twice as many as the first and the third three times as many as the first. How many will there be in each field?

15. A flock of 100 sheep is to be divided into two flocks so that one flock shall contain 12 more than the other. How many sheep will there be in each flock?

16. Twenty-seven crates can be loaded on two trucks. One truck holds twice as many as the other. How many crates will each hold?

John Wallis

17. Five hundred crates are to be packed by the workmen in two warehouses. Warehouse No. 1 has three times as many workmen as Warehouse No. 2. How many crates should be sent to each warehouse so that the work will be evenly divided among the men?

18. Three groups of men are employed to clean the streets after a snowstorm. The first group is twice as large as the second, and the third is as large as the first and second combined. There are 30 miles of street. How many blocks should each group clean if there are 10 blocks to a mile?

HINT. Let x denote the number of blocks that the second group can clean. Then $2x$ is the number of blocks that the first group can clean.

BIOGRAPHICAL NOTE

JOHN WALLIS. Among those who introduced and helped to standardize the modern algebraic symbolism was John Wallis, an Englishman. He was the son of a clergyman, and also took holy orders himself. Like most scholars of his day, he did not confine his interest to any one subject. He was at various times an instructor in Latin, Greek, and Hebrew, and for many years was professor of mathematics at Oxford. He also invented a method of teaching deaf mutes to talk.

During the wars between Charles I and Cromwell, Wallis's sympathies were with Cromwell, and he was of great service in reading royalist dispatches written in cipher. In fact, he was one of the most famous cryptologists of his day.

Wallis did not become interested in mathematics till the age of thirty-one, but devoted himself to the subject for the rest of his life. One of the earliest and most important books on algebra ever written in English was his treatise published in 1685. It contains a brief historical sketch of the subject, which is unfortunately not entirely accurate, but his treatment of the theory and practice of arithmetic and algebra has made the book a standard work for reference ever since.

REVIEW EXERCISES

1. Add $x^2 + xy - y^2$, $z^2 - yz - y^2$, $xz + z^2 - x^2$.

2. Add $x^2 + y^2 - 2\,xy$, $2\,z^2 - 4\,y^2 - 3\,yz$, $2\,x^2 - 4\,z^2 - 3\,xz$.

3. Add $3\,a^2 - 10\,b^2 + 5\,c^2 - 8\,bc$, $-a^2 + 4\,b^2 - 10\,c^2 - 3\,ab$, $c^2 + 11\,bc + 7\,ac - 5\,ab$, $4\,c^2 - 5\,bc + ac$, $-2\,a^2 + 6\,b^2 + 9\,ac + bc$.

4. Add $3\,x^2 + y^2 - 3\,yz + z^2$, $4\,xy + 3\,y^2 - 3\,yz$, $-2\,x^2 + 2\,xy - y^2 - 3\,z^2$.

5. Given $a = -5\,x + 3\,y + 2\,z$, $b = -5\,y + 3\,x + 2\,z$, and $c = -4\,z + 2\,x + 2\,y$, show that $a + b + c = 0$.

6. Given $x = b - 2\,c + 3\,a$, $y = -c - 2\,a - 3\,b$, and $z = -a + 2\,b + 3\,c$, show that $x + y + z = 0$.

7. From the sum of $4\,x^3 + 3\,x - 7$, $2\,x^2 - 3\,x - 3\,x^3 + 1$, and $-5\,x^3 - 2\,x + x^2 + 9$, subtract the sum of $2\,x^3 - 11\,x$ and $7\,x^3 - 5\,x^2 + 3 + 2\,x$.

8. The perimeter of a rectangle is 84 yards. The length is 6 yards more than twice the width. Find the length and the width.

9. The length of a rectangle is 4 feet less than five times the width. The perimeter is 88 feet. Find the dimensions.

10. A man is 8 years older than his wife; the age of their son is one fourth that of his mother. Their combined ages are 62 years. How old is each?

11. Give examples illustrating: (*a*) factors; (*b*) exponents; (*c*) coefficients.

12. Write five monomials; five binomials; five trinomials.

13. Simplify $4 + 7 - 3 + 4 \div 2 - 9 \div 3 + 6 \times 4 - 8 + 3$.

14. Simplify $10 + 3 \times 2 - 8 + 4 + 10 \div 5 + 18 + 6 \times 2 - 30 \div 2$.

15. What is the area of a rectangle if the length is twice the width, and the perimeter is 36 feet?

16. It is decided to build a circular track one-half mile long. What must be the radius of the circle? ($C = 2\pi r$, where $\pi = \frac{22}{7}$ and r is the radius.)

17. There are two numbers such that the first is five times the second, and the first is 32 more than the second. Find the numbers.

18. A father who is four times as old as his son is also 24 years older than his son. Find the age of each.

19. There are two numbers such that the larger is twice the smaller. If 12 is added to the larger, it will be six times as large as the smaller. Find the numbers.

20. A circular pond is being constructed in a certain park. Assuming that the bottom is level, how much water will it hold if the pond is 70 feet in diameter and is filled with water to the depth of 5 feet? What would it cost to fill it with water at ninety cents per 500 cubic feet? ($V = \pi r^2 h$.)

21. Find the horse power (H) of a 6-cylinder engine in which the diameter (d) of a cylinder is 5 inches, if the formula for the power of the engine is $H = d^2 n/6$, where n is the number of cylinders.

22. Some men standing on the edge of a perpendicular cliff wished to know how far it was to the stream of water at the foot. They dropped a rock over the cliff and saw the splash 4 seconds later. How high was the cliff? ($S = \frac{at^2}{2}$, where $a = 32$, and $t =$ the time in seconds.)

23. A certain rectangular playground is twice as long as it is wide. Its perimeter is 270 yards. Find the length and width.

CHAPTER VI

SUBTRACTION

37. Subtraction of monomials. In arithmetic it was found that subtracting 7 from 10 involved finding what number added to 7 would give 10. Similarly, to subtract $-2a$ from $+7a$, we find the number $+9a$, which when added to $-2a$ will give $+7a$.

EXAMPLES

1. From	$+10a$	3. From	$-10ay^4$
take	$+5a$	take	$+3ay^4$
Result	$+5a$	Result	$-13ay^4$
2. From	$7ax$	4. From	$-12x^2y^4$
take	$-3ax$	take	$-3x^2y^4$
Result	$10ax$	Result	$-9x^2y^4$

These examples illustrate the

Rule. *To subtract one number from another, change the sign of the subtrahend and add.*

EXERCISES

Subtract the lower number from the upper:

1. $\begin{array}{r} 10a \\ -3a \\ \hline \end{array}$

3. $\begin{array}{r} 10a^2xy^2z^3 \\ -10a^2xy^2z^3 \\ \hline \end{array}$

5. $\begin{array}{r} 10a^3 \\ -3a^3 \\ \hline \end{array}$

2. $\begin{array}{r} 4a^2xy^2 \\ -5a^2xy^2 \\ \hline \end{array}$

4. $\begin{array}{r} 7ay \\ -2ay \\ \hline \end{array}$

6. $\begin{array}{r} 3x^2y^4 \\ -5x^2y^4 \\ \hline \end{array}$

7. $\begin{array}{r} 6\,ax^2 \\ -\,3\,ax^2 \\ \hline \end{array}$ 9. $\begin{array}{r} -\,6\,ax^2 \\ -\,3\,ax^2 \\ \hline \end{array}$ 11. $\begin{array}{r} 6\,a^3y^4 \\ -\,8\,a^3y^4 \\ \hline \end{array}$

8. $\begin{array}{r} 4\,axy^4 \\ -\,axy^4 \\ \hline \end{array}$ 10. $\begin{array}{r} 17\,ayz \\ -\,12\,ayz \\ \hline \end{array}$ 12. $\begin{array}{r} -\,7\,ay^3 \\ 5\,ay^3 \\ \hline \end{array}$

Subtract:

13. $-3\,ax$ from $-7\,ax$.

14. $-6\,ax$ from $-2\,ax$.

15. $4\,ay^3$ from $-3\,ay^3$.

16. $-7\,ax^2$ from $3\,ax^2$.

17. $9\,ab^3$ from $-3\,ab^3$.

18. $-6\,a^2b^4$ from $3\,a^2b^4$.

19. $10\,axy^3$ from $4\,axy^3$.

20. $3(x+y)$ from $6(x+y)$.

21. $7(a-b)$ from $-4(a-b)$.

22. $-3(w+v)$ from $-2(w+v)$.

23. $-11(x^2-y^2)$ from $5(x^2-y^2)$.

24. $2\sqrt{x+3}$ from $3\sqrt{x+3}$.

25. $5\sqrt{x^2-y^2}$ from $-2\sqrt{x^2-y^2}$.

38. Subtraction of polynomials. Subtraction of one polynomial from another is illustrated in the following

EXAMPLE

Subtract $3\,x-4\,y-5\,z+4$ from $x+7\,y-4\,z$.

Solution.

$$\begin{array}{rrrrl} x & +\;7\,y & -\;4\,z & & = \text{the number from which something is subtracted, or the minuend.} \\ 3\,x & -\;4\,y & -\;5\,z & +\;4 & = \text{the number subtracted, or the subtrahend.} \\ \hline -\,2\,x & +\;11\,y & +\;z & -\;4 & = \text{the difference.} \end{array}$$

Check. Difference + subtrahend = the minuend. Adding the difference and the subtrahend in the solution above, we obtain $x+7\,y-4\,z$, which is the minuend.

This example illustrates the

Rule. *Write the subtrahend under the minuend, so that similar terms are in the same column.*

Then, changing mentally the sign of each term in the number to be subtracted, add the similar terms algebraically.

Check. *As in arithmetic, difference + subtrahend = minuend.*

EXERCISES

Subtract the first number from the second:

1. $a + 3,\ a + 4.$
2. $a - 4,\ a - 7.$
3. $10\,x + 7,\ 7\,x - 10.$
4. $3\,a,\ 3\,a - 2.$
5. $3\,a - 2,\ 3\,a.$
6. $4\,xy - b,\ 6\,xy.$
7. $6\,xy,\ 4\,xy - b.$
8. $7\,a^2x - 4\,y,\ 3\,a^2x.$
9. $6\,a^3,\ 5\,a - 7\,a^3.$
10. $5\,a - 7\,a^3,\ 6\,a^3.$
11. $4\,ab - 7,\ 0.$
12. $0,\ 4\,ab - 7.$

Subtract the first polynomial from the second and check the work:

13. $x + 2\,y - 3\,z,\ 3\,x + 2\,y + 2\,z.$
14. $3\,x - 7\,y + 3\,z,\ 3\,x - 4\,y - 5\,z.$
15. $2\,a - 3\,b,\ 5\,a + b + 3\,c.$
16. $3\,b - 4\,c + 5\,d,\ 3\,b - 5\,c.$
17. $2\,x^2 - 2\,x - 5,\ 3\,x^2 + 7\,x - 3.$
18. $a + x - y,\ b + x + y.$
19. $4\,a - 3\,b + 2\,c + 7,\ 5\,a - 2\,b + 6\,c.$
20. $2\,a - 2\,c - 4,\ a - 3\,b + 2.$
21. $3\,a - 4\,x + 5\,c,\ 5\,x - 4\,a - 10.$
22. $x + y - z,\ z - x - y.$
23. $3\,a^2 - 4\,ab + 2\,b^2,\ 4\,b^2 - 6\,a^2 + 5\,ab.$

24. $3a^3 - 3a^2b - 3ab^2$, $4a^2b - 5ab^2 - b^3$.

25. $c^3 - d^3 - 3c^2d + 3cd^2$, $5c^2d + 4d^3 + 3c^3 - 2cd^2$.

26. $3c^3 + 6cd^2 - 2a^2b - 6c^2d$, $3ab^2 - 3a^2b - 6c^2d + 6cd^2$.

27. $x^5 + 1 - 2x + 3x^4 + x^3 + 3x^2$, $6x^5 + 8x - 5x^2 + 16 + 6x^3 - 4x^4$.

28. $6x^5 + 2 - x^2 - 9x^4$, $4x - 3x^2 - 14 + 8x^5$.

29. $3x^2 + 9x^3 + 3 - x$, $7 + 5x^5 - x + x^3 - 2x^2 - 4x^4$.

30. Subtract the sum of $b^2 + 2bc - 1$ and $b^2 + 12bc - 20$ from $b^2 + 13bc - 30$.

31. Subtract the sum of $b - 3c + d$ and $4b + 5c - 6d + 4$ from $b - c + d - x$.

32. From the sum of $7x + 4x^2y - 16xy^2$ and $-8x - 13xy^2 + 8yx^2$ take $12x - 4x^2y + 8y^2x$.

33. From the sum of $5abc^2 - 4ab^2c + 3a^2bc$ and $5abc^2 - 4ab^2c - 3a^2b^2c$ subtract $2abc^2 - 4a^2bc + 8ab^2c$.

34. From the sum of $4x - 5xy - 3z$ and $8xy - 5z - 4x$ take the sum of $4z - 3xy - 2a^2bc$ and $9x - 5a^2bc - z$.

35. From the sum of $a^4 + a^2c^2 + c^4$ and $2c^4 - 3a^2c^2 - 4a^4$ take the sum of $2a^4 + 3c^4 + 3a^2c^2$ and $a^4 - 2c^4 - 4a^2c^2$.

36. From the sum of $3a + 5b - c$ and $3a - 5b + c$ take the sum of $2a - b + c$ and $b - c - 2a$.

37. From the sum of $7a^2 - 5a + 2$ and $5a - 7a^2 - 2$ take the sum of $3a^2 + 4a + 6$ and $-10 - 3a - 4a^2$.

38. From the sum of $3a + 4b - 3c$, $2a - 3b + c$, $-c - 2b - 3a$, and $4a - 3b - 4c$, take the sum of $a - b + c$, $b + c - 3a$, and $-c + 2a - 2b$.

39. From the sum of $6a^2y - 6ay^2 + y^3$, $a^3 - 3a^2y + 3ay^2$, and $3ay^2 - 3a^2y$, take the sum of $4a^2y + 3ay^2 + 2y^3$, $a^3 - 3ay^2$, and $-2a^2y - y^3$.

CHAPTER VII

IDENTITIES AND EQUATIONS OF CONDITION

39. Kinds of equations. By definition, an equation is the statement of equality between two equal numbers or number symbols (p. 12).

A literal equation in which the two members are alike, term for term, or can be made so, is called an *identity*.

An identity is true for any value of the letter or letters. An equation involving only numbers is always an identity.

Thus $2 + 4 = 3 \cdot 2$, $4x + 2 = 3x + 2 + x$, and $2x + x = 4x - x$ are identities. The latter equation is an identity, for, by combining terms, the equation reduces to $3x = 3x$.

An equation is called an *equation of condition* if it is true only for certain values of the unknown involved.

The equation $4x = x + 9$ is true when $x = 3$, for if 3 is substituted for x, the equation becomes $4 \cdot 3 = 3 + 9$, or $12 = 12$. Clearly the statement is false if 2, 5, or any value other than 3 is put for x. The equation $4x = x + 9$ is true on condition that x be 3, and on no other.

ORAL EXERCISES

State which of the following expressions are identities:

1. $5 + 2 = 8 - 1$.
2. $2x = 6$.
3. $3b - 5 = 2b$.
4. $x - 2 = 2x - 1$.
5. $2y = 4y - 3y + y$.
6. $a + 7 = 8 + a - 1$.
7. $3w - 1 + w + 1 - 4w = 0$.
8. $w + 4 - 3w + 2 = 6w - 8$.

In the following, select from the group at the right of each equation those numbers which satisfy that equation:

9. $4a - 12 = 0.$ 1, 2, 3.
10. $2b = b + 2.$ 1, 2, 4.
11. $d + 3 = 4d - 6.$ 0, -1, 3.
12. $3a - 4 = 2a + 1.$ -2, 1, 5.
13. $x + 6 = 3x - 10.$ 8, 5, -4.
14. $m + 1 - 2m + 3 = 0.$ 7, 8, -3.
15. $s^2 - 4s + 4 = 0.$ 2, 0, -2.
16. $2a^2 - 8a + 8 = 0.$ 2, -2.
17. $x^3 - 9x^2 + 27x - 27 = 0.$ 3, -3.
18. $a^2 - 5a + 6 = 0.$ 2, 3, 0.

40. Root of an equation. A number or a literal expression is called a *root of an equation* if, after substituting it for the unknown letter in the equation, the result is or can be reduced to an identity.

Thus 3 satisfies the equation $4x = x + 9$ and, similarly, $2a$ satisfies the equation $x - 5 = 2a - 5$.

A number or number symbol is called a root of an equation if it satisfies the equation.

ORAL EXERCISES

Is the number at the right of each of the following equations a root of that equation?

1. $3a - 24 = 0.$ 8.
2. $4x - 9 = x.$ 3.
3. $5r - 12 = r.$ 2.
4. $a^2 + 5a - 6 = 0.$ 4.
5. $d^2 - 16 = 4d + 5.$ 7.
6. $m^2 + 7 = 6m - 1.$ 4.
7. $a^2 + 5a + 4 = 0.$ -2.
8. $m + 3 = 2m - 1.$ -3.

41. Transposition. In solving the equation $5x - 2 = 18$ we add 2 to each member. If we indicate this addition, the equation becomes

$$5x - 2 + 2 = 18 + 2.$$

In the first member, $-2 + 2 = 0$. Hence these two numbers may be omitted, and the equation becomes

$$5x = 18 + 2.$$

Comparing this with the original equation,

$$5x - 2 = 18,$$

we see that -2 has vanished from the first member of the original equation and $+2$ has appeared in the second member of the new equation. The number $+2$ has really been added to both terms of the equation.

Again, if we subtract $2y$ from both members of

$$6y = 2y + 12,$$

we get the equation $6y - 2y = 12$,

which differs from the original equation only in having $-2y$ in the first member instead of $+2y$ in the second. Hence a term may be omitted from one member of an equation if the same term with its sign changed from + to − or from − to + is written in the other member.

This process is called *transposition.*

Hereafter, instead of going through the details of subtracting a number from both members of an equation (or adding a number to both members), the student should use transposition. He should remember, however, that the transposition of a term is really the subtraction of that term from (or addition to) each member of an equation.

Like terms in the same member of an equation should be combined before transposing any term.

NOTE. Our word *algebra* is derived from the Arabic word for *transposition.* The process by which one passes from the equation $px - q = x^2$ to the equation $px = x^2 + q$ was known as *al-jebr.* This is the first word in the title of an Arabic book on algebra which was translated into Latin. For some reason only this part of the title remained, and by the early part of the seventeenth century *al-jebr*, or algebra, was the common name given to the whole subject.

ORAL EXERCISES

State the term or terms which must be added to both members of the following equations in order to transpose the underscored terms. What does each equation become after the operation is performed?

1. $x - \underline{5} = 3.$
2. $n + \underline{12} = 3.$
3. $x - \underline{4} = 2.$
4. $n^2 + 6n = \underline{4n} - 3.$
5. $x^2 + 4x = \underline{5}.$
6. $a^2 + 3a + 2 = \underline{2a - 3}.$
7. $3a + \underline{3} + 4a + \underline{5} = \underline{3a} + 4.$
8. $5x - \underline{3} - 4x + \underline{7} = 8 - \underline{2x}.$
9. $r^2 + 2r = -\underline{4}.$
10. $3d - 2 = -\underline{2d}.$
11. $5x - 8x = \underline{5x} - \underline{2}.$
12. $6r - r^2 = \underline{2} - \underline{2r^2}.$

EXAMPLE

Solve the equation

$$32n - 13 + 7n + 9 - 4n = 7n + 16 + 12n + 12.$$

Solution. $32n - 13 + 7n + 9 - 4n = 7n + 16 + 12n + 12.$

Combining like terms, $35n - 4 = 19n + 28.$

Transposing, $35n - 19n = 28 + 4.$

Combining like terms, $16n = 32.$

Dividing by 16, $n = 2.$

Check. $32n - 13 + 7n + 9 - 4n = 7n + 16 + 12n + 12.$

Substituting 2 for n,

$$64 - 13 + 14 + 9 - 8 = 14 + 16 + 24 + 12.$$

Combining, $66 = 66.$

EXERCISES

Solve the following equations and check:

1. $4x - 2 = 3x + 2$.
2. $3x + 5 = x - 5$.
3. $-7y - 2 = 1 - 10y$.
4. $6x + 3 - 3x = 21$.
5. $3y + 5 = 6 + 2y$.
6. $4h + 1 - 2h = 5$.
7. $16 + 4h - 4 = h + 12$.
8. $4h - 15 = 15 - h - 5$.
9. $5h + 11 - 2h = 6 - 4$.
10. $2h - 1 = 29 + 7h$.
11. $3h + 9 + 5h - 33 = 0$.
12. $3x - 20 + 8x - 24 = 0$.
13. $6h - 19 + 2h + 3 = 0$.
14. $h + 2 - 5h = 9h + 15$.
15. $4h + 3 = 15 + 2h + 6$.
16. $3h + 5 + h + 3 = 0$.
17. $9 - 8n + 2 = 3 - 4n$.
18. $3 - 5n + 2 = 5 + 7n$.
19. $8n - 3n = 15n + 4 - 13n$.
20. $8 + 7n - 13 = n - 27 - 5n$.
21. $3n - 15 - 10n - 9 + 16n - 21 = 0$.
22. $18 + 5n - 6 - 2n + 1 + 3n - 25 = 0$.
23. $0 = 18 - 4n + 27 + 9n - 3 + 16n$.
24. $5t - 8 + 4t + 5 = 7t - 3 - 2t + 5$.
25. $0 = 4p - 15 - 11p - 18 + 16p - 17$.
26. $5p - 6 + 3p + 18 - 2p - 25 + 1 = 0$.
27. $3p + 18 - 2p - 3 = 7p - 5 - 4p + 8$.
28. $0 = 6 - 8p + 8 - 2 + 4p$.
29. $3p + 5 + 8p + 50 = 0$.
30. $2 - 2x + 3x - 3 = 2x - 2$.
31. $2d + 3 + 3d + 5 = 4d + 6$.
32. $2a^2 + 3a + 4 = 2a^2 + 2a$.
33. $3s^2 + 12s = 3s^2 + 6s - 12$.
34. $a + 4 + a^2 + 2a - 6 = a - 3 + a^2$.
35. $n^2 - n + 1 = n^2 + n - 1$.
36. $r^4 - r + 6 = r^4 + r - 6$.

ORAL EXERCISES

Represent a number:

1. Greater by 5 than 4.
2. Greater by 5 than x.
3. Greater by 5 than y.
4. Greater by x than y.
5. Greater by x than twice y.
6. Nine more than y.
7. Nine more than $x + y$.
8. Two less than 4.
9. Four less than x.
10. Four less than $x + y$.
11. Less by 7 than x.
12. Less by y than x.
13. Less by y than twice x.
14. Less by y than three times x.
15. Three times p.
16. One half as great as p.
17. Three more than half of p.
18. p less than three times x.
19. p less than $x + y$.
20. p more than y less z.
21. One part of 10 is 6. What is the other part?
22. One part of x is 6. What is the other part?
23. One part of x is 10. What is the other part?
24. One part of n is a. What is the other part?
25. One part of a is n. What is the other part?
26. One part of $n + p$ is 1. What is the other part?
27. The sum of two numbers is 15. One is 10. What is the other?
28. The sum of two numbers is x. One of them is 3. What is the other?
29. The difference between two numbers is 5. The less number is 3. What is the other?

30. The difference between two numbers is 12. The less number is 17. What is the other?

31. The sum of two numbers is n. One number is p. What is the other?

32. The sum of two numbers is n. One of the numbers is three times as large as p. What is the other?

33. The difference between two numbers is 12. The greater number is 17. What is the less?

34. The difference between two numbers is x. The greater number is y. What is the less?

35. The difference between two numbers is 17. The less number is 4. What is the greater?

36. The difference between two numbers is x. The less number is p. What is the greater?

37. By how much does 12 exceed 8? 15 exceed 10? 17 exceed 11? x exceed 3? x exceed 10? x exceed y?

38. How much less is 6 than 10? x than 20? x than y?

39. By how much does $n + 6$ exceed 6? $n + 6$ exceed p? $3n + 6$ exceed $2n$? $5n - 3$ exceed $4n$?

42. Translation of problems into equations. In the solution of problems the writing of the equations is merely translating the statement of the problem from ordinary language into the language of algebra. Sometimes it is possible to translate the original statement of the problem word by word into algebraic symbols.

For example:

Three times a certain number, diminished by 6,
$3 \quad \times \quad n \quad - \quad 6$
gives the same result as the number increased by 14.
$= \quad n \quad + \quad 14$

PROBLEMS

Solve the following problems, making direct translations into the symbols of algebra whenever possible:

1. What number increased by 12 is equal to 30?

2. What number increased by 20 is equal to 30?

3. What number increased by 10 is equal to twice the number?

4. What number diminished by 20 is equal to 9?

5. What number diminished by 12 is equal to twice the number?

6. To what number must 20 be added so that the result may be 30?

7. To what number must 20 be added so that the result will be 15?

8. From what number must 10 be subtracted so that the result may be three times the number?

9. A certain number is doubled and the result increased by 10. The sum is 14. What is the number?

10. Three times a certain number less 25 equals twice the number less 15. What is the number?

11. Five times a certain number increased by 14 equals eight times the number diminished by 4. Find the number.

12. Four times a certain number increased by 7 equals five times the number diminished by 7.

13. What number is as much less than 80 as it is greater than 26?

14. What number is as much less than 100 as it is greater than 50?

15. What number exceeds 20 by twice as much as 50 exceeds the number?

16. A certain number added to 17 gives the same result as that obtained when twice the number is subtracted from 62. What is the number?

17. Twice a certain number added to 30 gives the same result as three times the number subtracted from 90. What is the number?

18. The sum of two numbers is 54, and their difference is 12. What are the numbers?

Solution. Two numbers are to be found. Their sum is 54, their difference 12.

Greater number + less number = 54.

If $n =$ less number,

then $n + 12 =$ greater number.

Placing these symbols in the principal statement above, we have

$$n + 12 + n = 54.$$

Combining, $2n + 12 = 54.$

Transposing, $2n = 54 - 12.$

Combining, $2n = 42.$

Dividing by 2, $n = 21,$

and $n + 12 = 33.$

Check. $21 + 33 = 54$, $33 - 21 = 12$.

19. The sum of two numbers is 75, and their difference is 41. What are the numbers?

20. The sum of two numbers is 14, and their difference is 30. What are the numbers?

21. The sum of two numbers is 50, and their difference is three times the smaller number. Find the numbers.

22. The sum of two numbers is 10. The greater is 14 more than the less. What are the numbers?

23. The sum of two numbers is 57, and one exceeds the other by 15. What are the numbers?

24. The sum of three numbers is 115. The second is 7 greater than the first, and the third is 8 greater than three times the first. Find the numbers.

25. The sum of three numbers is 37. The first is 5 less than the second, and the third is 2 more than twice the second. What are the numbers?

26. A triangle whose perimeter is 45 inches has sides in the ratio of 2, 3, and 4. Find the sides.

27. The first side of a triangle is 3 inches less than the second, and the third is 2 inches more than twice the first. The perimeter is 37 inches. What are the sides?

28. A rectangle whose perimeter is 36 feet is twice as long as it is wide. What are its dimensions?

29. A rectangle whose perimeter is 56 feet is three times as long as it is wide. What are its dimensions?

30. A certain garden requires 320 yards of fencing for the four sides. The garden is four times as long as it is wide. Find the length and width of the garden.

31. A real estate dealer can afford to spend only $80 per week for office help. If the weekly salary of the office boy is $10 and that of the stenographer is three fourths as much as that of the bookkeeper, how much should he pay each?

32. A man can allow his three children all together $2 a week for spending money. James needs 50 cents per week more than Ruth, and Charles requires only half as much as Ruth. What allowance will each child receive?

33. Mr. James paid \$12 for tickets for himself and his wife and half-fare tickets for John, aged 10, and Frank, aged 7. What is the price of one full-fare ticket?

34. Find two consecutive numbers whose sum is 53.

35. Find three consecutive numbers whose sum is 66.

36. Find four consecutive numbers whose sum is 74.

37. Find two consecutive odd numbers whose sum is 92.

HINT. Let $n =$ the first number, and $n + 2$ the second number.

38. Find two consecutive odd numbers whose sum is 108.

39. Find three consecutive odd numbers whose sum is 117.

40. Find four consecutive odd numbers whose sum is 192.

41. Find two consecutive even numbers whose sum is 46.

42. Find three consecutive even numbers whose sum is 108.

43. Find five consecutive even numbers whose sum is 60.

CHAPTER VIII

PARENTHESES

43. Removal of parentheses. In solving exercises and problems it is often necessary to treat several terms as though they were one term. This is done by inclosing the various terms in a parenthesis.

Thus, if n represents an even number, the sum of three consecutive even numbers might be represented as follows:

$$n + (n + 2) + (n + 4).$$

Sometimes it is necessary to inclose a number in a parenthesis along with other terms inside a second parenthesis. To avoid confusion in such cases, different parentheses, such as brackets [ ] and braces { }, are used. Parentheses, brackets, and braces are sometimes called symbols of aggregation. For convenience, where no confusion arises, braces and brackets are spoken of as parentheses.

In the solution of equations and in other algebraic work it is often necessary to remove all signs of grouping. This removal, although it depends upon the principles governing addition and subtraction, must be given some special study if the required speed and accuracy are to be attained.

44. Removal of parentheses preceded by the sign +. The value of $10 + (4 - 1)$ is the same as that of $10 + 4 - 1$. Similarly, $a + (b - d) = a + b - d$.

The plus signs preceding the parentheses $(4-1)$ and $(b-d)$ in the above expressions disappear with the parentheses, but the signs of the terms within the parentheses, whether expressed or understood, are always supplied when the parentheses are removed, as in the case of 4, 1, b, and d, in the above expressions. Since in adding one expression to another the signs of the terms added are not changed, we have the following

Principle. *A parenthesis and the plus sign before it may be removed from an expression without changing the signs of the terms which were inclosed by the parenthesis.*

45. Removal of parentheses preceded by the sign –. In the expression $10-(5-3)$ the sign before the binomial shows that $(5-3)$ is to be subtracted from 10. To subtract $(5-3)$ we change the signs of the terms subtracted and add the resulting terms to the minuend.

Thus, $10-(5-3)=10-5+3=8$. This is correct, since $10-(5-3)=10-2=8$.

Similarly, $a-(c-d)$ becomes $a-c+d$ when the signs of $(c-d)$ are changed and the result is added to a.

The minus signs preceding the parenthesis in $10-(5-3)$ and $a-(c-d)$ disappear when the parentheses are removed. The plus signs understood before 5 and c in the parentheses are changed, as is the sign of each term within the parentheses, when we write $10-(5-3)=10-5+3$, and $a-(c-d)=a-c+d$. Since, in subtracting one expression from another, we add with their signs changed the several terms to be subtracted, we have the following

Principle. *A parenthesis and the minus sign before it may be removed from an expression, provided the sign of each term which was inclosed by the parenthesis is changed.*

ORAL EXERCISES

Remove the parentheses and where possible combine terms in the resulting expressions:

1. $7 + (4 - 1)$.
2. $7 - (4 + 1)$.
3. $7 - (-4 + 1)$.
4. $x + [w + v]$.
5. $x + \{w - v\}$.
6. $x - (w + v)$.
7. $x - \{w - v\}$.
8. $x + (-w - v)$.
9. $x - [-w - v]$.
10. $3x - (x - 2y + 5)$.

46. Removal of two or more parentheses. If neither of two parentheses is within the other, both parentheses may be removed at the same time, proper regard being given to the principles governing the removal of each.

If one parenthesis incloses another, the inner parenthesis should be removed first, in accordance with the following

RULE. *Rewrite the expression, omitting the innermost parenthesis, changing the signs of the terms which it inclosed if the sign preceding it is minus and leaving them unchanged if it is plus.*

Combine like terms within the new innermost parenthesis.

Repeat these processes until all the parentheses are removed.

EXAMPLE

Remove the parentheses and collect terms:

$$12 - [15 - 8a + (18 - 9a)] + 2.$$

Solution. Removing the inner parenthesis first,

$$12 - [15 - 8a + 18 - 9a] + 2.$$

Collecting terms, $12 - [33 - 17a] + 2$.

Then removing the remaining parenthesis,

$$12 - 33 + 17a + 2.$$

Collecting terms, $17a - 19$.

EXERCISES

Remove the parentheses and combine like terms:

1. $10 - (6 - 3) - 4$.
2. $12 + (7 - 4) - (8 - 6)$.
3. $(8 - 5 + 2) - (4 - 2) + 7$.
4. $10a - (6a - 3a) + (7a - a)$.
5. $(2n - 5p) - (2n - p - 6n)$.
6. $n - (p - q) + \{2q - 3p\}$.
7. $n - p - (m - q) + (n - p) - (m - n)$.
8. $(n - p) - \{2p - 3n\} + (n - 4p)$.
9. $n - (n - p - q) - (3q + p + 1) + (n - 5)$.
10. $(n - p) - (3n - 2p) + [2p - n] - 3p$.
11. $8 - [9 - (4 - 10)] - (15 - 24)$.
12. $9 - [5 - (4 - 6)] - (8 - 15)$.
13. $14 - (8 - 16) - [(4 - 1) - 8]$.
14. $15 - [3 - (2 - 5)] + (3 - 5) - (8 - 3)$.
15. $p - [2n - (3p - 5)]$.
16. $[(n + p) - n] - p$.
17. $[(a + p) + (a - 5p)] - 4a$.
18. $(-r - s) - [(2r - s) + (r + s)]$.
19. $-(-a - p) - [(-a + 5p) + (-2p - a)]$.
20. $a + [3a - (4a - 2b)] + (3b - 2a)$.
21. $a + [4a - (3x - 2a)] - (4a - 5x)$.
22. $2a - [6a - (5n - 4a)] + (8a - 7n)$.
23. $(5x - 6y) - [-4x - (4z - y)] - 2z$.
24. $[3p - (2n + q)] - [-(3n - 2p) + 5p]$.

In the following, remove the parentheses, retaining the brackets, and simplify results as much as possible:

25. $[(n + p) + r], [(n + p) - r]$.
26. $[4\,n + (3\,r - 5\,p)], [4\,n - (3\,r - 5\,p)]$.
27. $[(n - p) + (p - 2\,n)], [(n - p) - (p - 2\,n)]$.
28. $[(3\,n - 2\,p) + (2\,p - 3\,n)], [(3\,n - 2\,p) - (2\,p - 3\,n)]$.
29. $[(n - 2\,p) + (3\,r - s)], [(n - 2\,p) - (3\,r - s)]$.
30. $[(4\,n - 3) + (5\,p - 7)], [(4\,n - 3) - (5\,p - 7)]$.
31. $[(n^2 - p^2) + (r^2 - 2\,p^2)], [(n^2 - p^2) - (r^2 - 2\,p^2)]$.

47. Inclosing terms in parentheses. Obviously, $12 + 7 - 3 = 12 + (7 - 3)$, for each equals 16.

Similarly, $c + d - f = c + (d - f)$.

That the expressions in the illustration above are equal may be seen by removing the parentheses according to the first principle on page 84.

Thus, $12 + (7 - 3) = 12 + 7 - 3$, and $c + (d - f) = c + d - f$.

This process of inclosing terms within a parenthesis illustrates the

PRINCIPLE. *One or more terms may be inclosed in a parenthesis preceded by a plus sign without changing the sign of any of the terms.*

ORAL EXERCISES

Inclose in a parenthesis preceded by a plus sign the last three terms in each of the following:

1. $4 + 7 - 6 - 3$.
2. $8 + 4 + 7 - 2$.
3. $9 - 4 - 7 + 3$.
4. $a - b + c + d$.
5. $a + b - c - d$.
6. $3\,a + 4\,b - c - d$.
7. $3\,n - 4\,p - q - 3\,n$.
8. $n^2 - 2\,p + p^2 + 1$.

The expression $15 + 7 - 4 = 15 - (-7 + 4)$, for each equals 18.

Similarly, $x + y - z = x - (-y + z)$,

and $x - y + z = x - (y - z)$.

That the right member in each of these cases is another form of the left may be seen by removing the parentheses according to the second Principle on page 84.

Thus, $15 - (-7 + 4) = 15 + 7 - 4$, the original expression, and $x - (y - z) = x - y + z$.

This process of inclosing terms in a parenthesis preceded by a minus sign illustrates the

Principle. *One or more terms may be inclosed in a parenthesis preceded by a minus sign provided the sign of each term thus inclosed be changed.*

EXERCISES

Inclose in a parenthesis preceded by a minus sign the last three terms in the following:

1. $4 - 7 - 6 - 3$.
2. $7 + 3 + 6 - 2$.
3. $8 - 4 - 7 + 2$.
4. $a - b + c + d$.
5. $a - b - c - d$.
6. $3p + 4n - q - r$.
7. $3p - 4n + q - 3r$.
8. $n^2 + 2p - p^2 - 1$.
9. $p^2 - 4n - n^2 - 4$.
10. $p^2 - 2np + n^2 - n^2 + n + 1$.
11. $p + q - r - 3n + 4m$.
12. $p + q + r + n + m$.
13. $2p - q + 3r + 2p - q - r + qr$.
14. $4p^2 - 4q^2 + n^2 + 2pn + p^2$.

In the following, inclose in a parenthesis preceded by a plus sign all the terms containing x or y, and inclose in a parenthesis preceded by a minus sign all the other terms:

15. $x^2 - a^2 - 3a$.
16. $10a + x^2 - 25 - a^2$.
17. $y^2 - b^2 + 2ab - a^2$.
18. $20ab + x^2 - 4a^2 - 25b^2$.

19. $x^2 - b^2 - 4b^2 + 2y^2 - a^2 - 2xy - y^2$.

20. $-4ab + x^2 - b^2 + 4y^2 - 4a^2 - 4xy$.

21. $4x^2 - a^2 - 8xy - 4 + 4a + 4y^2$.

22. $x^2 - 9b^2 - 10xy + 36ab - 36a^2 + 25y^2$.

23. $4ab - 4a^2 + x^2 + 4xy - b^2 + 4y^2$.

24. $x^2 - 16xy - p^2 + 16p + 6xy^2 - 6x$.

25. $x^2 + a^2 + 2ab + 8xy + 2bc + 16y^2 - 2ac - b^2 - c^2$.

REVIEW EXERCISES

In the first four exercises, is the number to the right a root of the corresponding equation?

1. $x^2 - 7x + 6 = 2x + 4 - (8 + 2x)$. 2.

2. $a^2 - (2a - 3) = a^2 - (7 - 2a) + 6$. 3.

3. $(4-2x)-(3x-12)+(4x-10)=4x-20-(1-4x)$. 3.

4. $(3x + 6) - (4 - 7x) = 19x - (11x - 4)$. 7.

5. Indicate the sum of $2x$ and $(x + 3)$; 3 and $(x - 5)$; $(7 + x)$ and $6x - 2$.

6. Indicate the difference found by subtracting $(4x - 7)$ from $(6x + 3)$.

7. Indicate $(9x + 2)$ diminished by $(3x - 1)$ plus $(2 - 5x)$.

8. Indicate the sum of $(3x + 2)$, $(5x + 6)$, and $(9x - 3)$ diminished by the sum of $(4x - 7)$ and $(3x + 2)$.

9. Indicate the ages of three boys if the first is twice as old as the second and the third 3 years younger than the second. Write the equation if the sum of their combined ages is 21.

10. Find three numbers whose sum is 34, if the first two are consecutive even numbers and the third is twice the first.

CHAPTER IX

MULTIPLICATION

48. Product of terms containing unlike letters. The student is familiar with the fact that the factors of a product may be taken in any order.

Thus, $3 \cdot 2 \cdot 4 = 4 \cdot 3 \cdot 2 = 2 \cdot 3 \cdot 4 = 24.$

Similarly, $a \cdot b = b \cdot a.$

Again, $2\,m^2 \cdot 3\,n^2 = 3\,m^2 \cdot 2\,n^2 = 2 \cdot 3\,m^2n^2 = 6\,m^2n^2.$

This illustrates the

RULE. *To multiply terms containing numbers and letters, write the product of the numeric coefficients followed by the product of all the literal factors.*

49. Product of terms containing like letters. By the definition of an exponent (p. 17), $x^2 = x \cdot x$, and $x^3 = x \cdot x \cdot x$.

Therefore $x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x) = x^5 = x^{2+3}.$

Similarly,

$$a \cdot a^3 \cdot a^5 = a \cdot (a \cdot a \cdot a) \cdot (a \cdot a \cdot a \cdot a \cdot a) = a^9 = a^{1+3+5}.$$

In like manner,

$$3^2 \cdot 3^4 \cdot 3^5 = (3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)$$
$$= 3^{11} = 3^{2+4+5}.$$

Also, $np^2 \cdot p^3 = np^5 = np^{2+3},$

$$3\,ab \cdot 2\,a^2 = 6\,a^3b = 6\,a^{1+2}b,$$

and $3\,x^2yz^2 \cdot 5\,xy^3 = 15\,x^3y^4z^2 = 15\,x^{2+1}y^{1+3}z^2.$

Therefore we have the

Principle. *The exponent of any letter in a product is equal to the sum of the exponents of that letter in the factors. This is expressed in general terms thus:*

$$n^a \times n^b = n^{a+b}.$$

The law of signs for the multiplication of positive and negative numbers, given on page 41, applies to literal terms as well.

Thus,
$$(+3n^2)\cdot(+5n^3) = +15n^5,$$
$$(+3n^2)\cdot(-5n^3) = -15n^5,$$
$$(-3n^2)\cdot(+5n^3) = -15n^5,$$
$$(-3n^2)\cdot(-5n^3) = +15n^5.$$

For the multiplication of two monomials, we have the

Rule. *Obeying the rule of signs for the multiplication, write the product of the numeric coefficients followed by all the letters that occur in the factors, each letter having as its exponent the sum of the exponents of that letter in the factors.*

ORAL EXERCISES

Perform the following indicated multiplications:

1. $(3)(-5)$.
2. $(-2)(6)$.
3. $(-7)(-3)$.
4. $(-4x)(3)$.
5. $3x\cdot 2x$.
6. $n^2\cdot n^3$.
7. $n^2\cdot n^5$.
8. $n^4\cdot n^5$.
9. $n^2\cdot n^2$.
10. $n^3\cdot n^3$.
11. $n^5\cdot n^7$.
12. $-n^2\cdot(n^4)$.
13. $-n^3\cdot(-n^2)$.
14. $-n^2\cdot(+3n^3)$.
15. $-n(2n^5)$.
16. $(-2p^2)(3p^3)$.
17. $(3p^4)(-2p^5)$.
18. $(4p^2)(-5p^4)$.
19. $(-3p^3)(-4p^4)$.
20. $(-6b^2)(4b)$.
21. $(-3x)(-2x^3)$.

22. $(+2a^2)(-4a^2x)$.
23. $(-4a^3x)(+3x^3)$.
24. $(-5a^2x)(-4ax^2)$.
25. $(+3a^2r^3)(3ar)$.
26. $(-3n)^2$.
27. $(5)(-7a)$.
28. $(3a)(-7)$.
29. $(-2n)^2$.
30. $(-7a)(-10)$.
31. $(-3np)^2$.
32. $(3a)(-7a)$.
33. $(5npq)^2$.
34. $(-10x)(4x)$.
35. $(5x)(-4x)$.
36. $(-n)^2$.
37. $(-n)^3$.
38. $(-2n)^2$.
39. $(-2n)^3$.
40. $(2a^2x)^2$.
41. $(-2n)(-4n^2)$.
42. $(5m^4)(9m^3)$.
43. $(-5p)^3$.
44. $(n^8)(-16n)$.
45. $(-4n^5)(-6n^3)$.
46. $(+3p)^2$.
47. $(+3p)^3$.
48. $(4p)(5n)(6np)$.
49. $(3p)(-2n)^2$.
50. $(-3n^2p)^2$.
51. $(3h^2)(-k)$.
52. $(5n^2p)(-2n^3)$.
53. $(-5x^3y^2)^2$.
54. $(-5x^3y^2)^3$.
55. $(-n^4p)(-n^2p^4)$.
56. $(2np^2)^3$.
57. $(5n^3)(-4n^2)(-3n)$.
58. $(-2k^2)(3k^3)^2$.
59. $(-2a^2)^3(5a^3)^2$.

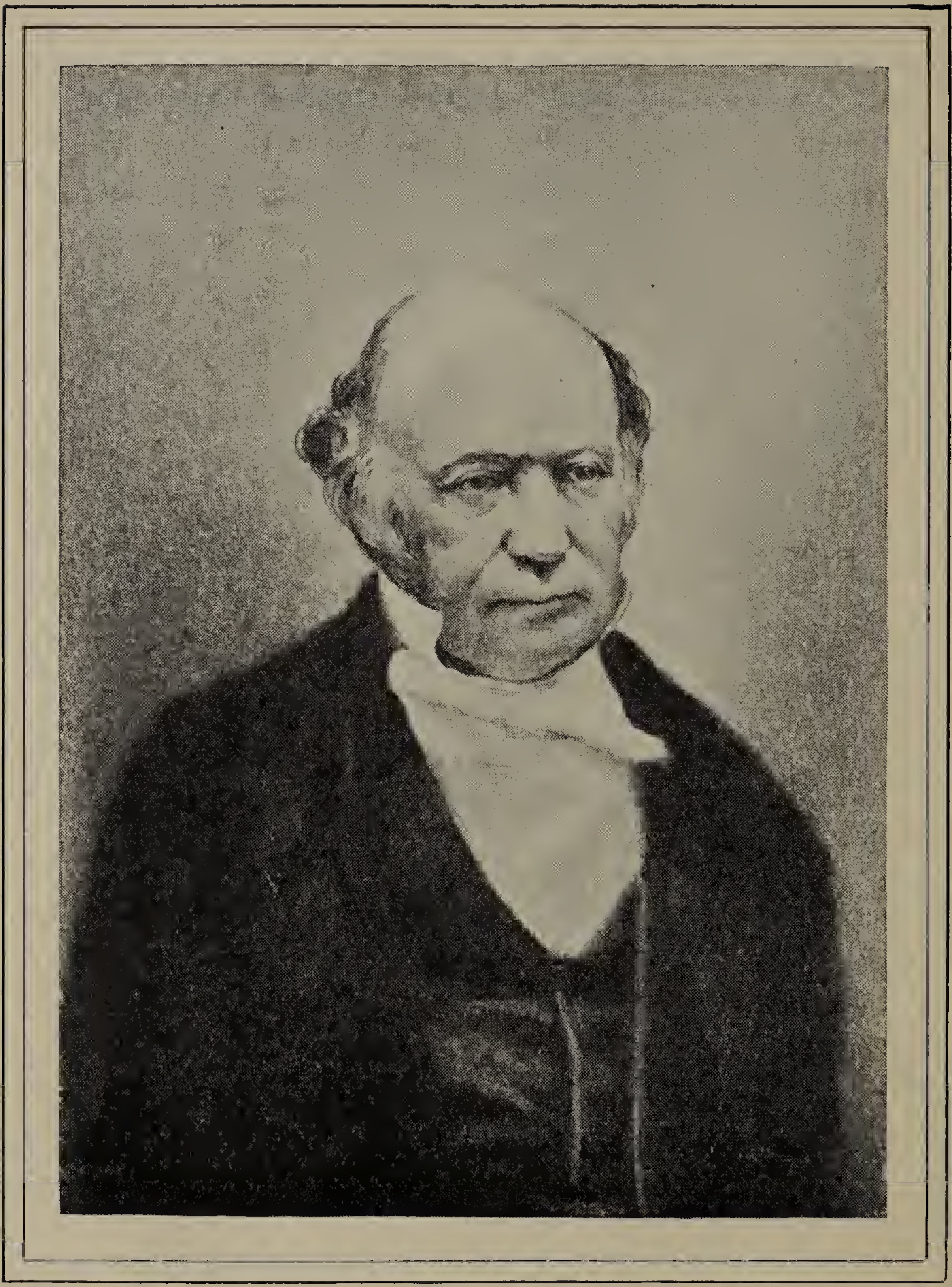
50. Multiplication of a polynomial by a monomial. The multiplication of a polynomial by a monomial may be illustrated as follows:

$$4(5+3) = 4 \times 5 + 4 \times 3 = 20 + 12 = 32,$$

and $$3(6+4) = 3 \times 6 + 3 \times 4 = 18 + 12 = 30.$$

Similarly, $$b(x+y) = bx + by,$$

and $$2a(x+y) = 2ax + 2ay.$$



Sir William Rowan Hamilton

For the multiplication of a polynomial by a monomial, we have the

Rule. *Multiply each term of the polynomial by the monomial and write the resulting terms in succession with their proper signs.*

The actual work of multiplication may conveniently be arranged as in the following

Example

Multiply $2n^2 - 3ny + 4y - 3p - 6$ by $2np$.

Solution.

$$\begin{array}{r} +2n^2 - 3ny + 4y - 3p - 6 \\ +2np \qquad\qquad\qquad\qquad\qquad \\ \hline 4n^3p - 6n^2py + 8npy - 6np^2 - 12np \end{array}$$

BIOGRAPHICAL NOTE

Sir William Rowan Hamilton. It is strange that of all the topics treated in this book the last to be thoroughly understood by mathematicians are those appearing in the first chapters. But in all the sciences it is often most difficult to answer the questions that at first sight seem quite obvious. Any child can ask what electricity is, but the wisest scientist cannot tell. He can only explain what electricity does. It is easy to ask how the earth came to be revolving around the sun with the moon revolving around it, but even the deepest students of astronomy differ in their theories of how it came to be. And so in mathematics, long after many of the more complicated processes of algebra were completely understood, the simple laws of operation of numbers were only slightly understood. One of the men who did most to clarify the nature of these laws was Sir William Rowan Hamilton (1805–1865). He was born in Dublin, Ireland, where he lived most of his life. He was a precocious boy, and at the age of twelve was familiar with thirteen languages. He devised kinds of numbers that do not follow the same laws as those that we use in algebra, and so threw a flood of light on the nature and properties of these common numbers. Most of his works are very advanced in character and are difficult to read.

EXERCISES

Multiply:

1. $x + 2$ by 3.
2. $x - 3$ by x.
3. $x^2 + 4$ by $3x$.
4. $2x^2 + 5$ by $2x$.
5. $x^2 - 3x$ by x^2.
6. $n^2 - 3n + 2$ by $3n$.
7. $p^3 - 5p^2 + 3p - 1$ by p^2.
8. $4n^2 - 2n - 4$ by n^2.
9. $-4p^2 + 6p - 5$ by $6p^2$.
10. $2p^2 - 3p - 3$ by $-4p^2$.
11. $p^3 - 3p^2 - 4$ by $-5p^4$.
12. $3x - 4x^2 - 3x^3$ by $-3x^3$.
13. $7xy^2 - x + y$ by $2xy$.
14. $x^2 - 2xy + 4y^2$ by $-2xy$.
15. $n^4 - n^2p^2 + p^4$ by $-n^2p^2$.
16. $-n^2p^2 - 2np + 7q^2$ by $-4npq$.
17. $8x^3 - 7x^2 + 11x - 5$ by $-3x^3$.
18. $-8a^2 - 10ax + 40x^2$ by $5ax^3$.

Perform the indicated multiplications:

19. $8(2x - 3)$.
20. $4x(x - y)$.
21. $-7(3x - 6)$.
22. $-8(-3a + 2b)$.
23. $-2x(3x - 6)$.
24. $5x^2(8x^3 - 3x)$.
25. $-3(n^2 - 2n - 6)$.
26. $5np(p^2 - 6p + 9)$.
27. $-2n(np - rp - 3sp^2)$.
28. $(n^3 - an + a^2)(-3a^2n)$.
29. $-6bc(bx^2 - cx + d)$.
30. $3x^2(-2x + 3x^2 - 2x^4)$.

31. What is the area of a square whose side is x inches? $2x$ inches? $3x$ inches? $4x$ inches?

32. What is the area of a rectangle whose length is $3x$ and whose width is $2x + 3$?

33. What is the volume of a rectangular solid whose dimensions are $x + 1$, $2x$, and $5x$?

34. The edge of a certain cube is $3x$ inches. What is the total area of the cube? its volume?

51. Multiplication of polynomials. Clearly,

$$(6+3)(7+4)=9\times 11=99.$$

The multiplication may also be performed thus:

$$\begin{aligned}(6+3)(7+4)&=6(7+4)+3(7+4)\\&=42+24+21+12=99.\end{aligned}$$

In general terms,

$$(a+c)(b+d)=a(b+d)+c(b+d)=ab+ad+bc+cd.$$

EXAMPLE

Multiply $2x-5$ by $3x+2$.

$$\begin{array}{lrl} \textit{Solution.} & 2x-5 & \\ & 3x+2 & \\ \cline{2-2} \text{Multiplying by } 3x, & 6x^2-15x & =\text{first partial product.} \\ \text{Multiplying by } 2, & +4x-10 & =\text{second partial product.} \\ \cline{2-2} \textit{Complete product,} & 6x^2-11x-10 & =\text{sum of partial products.} \end{array}$$

This gives for the multiplication of polynomials the

RULE. *Multiply each term of the multiplicand by each term of the multiplier in turn, and add the partial products.*

EXERCISES

Multiply:

1. $x+3$ by $x+2$.
2. $2x+4$ by $x+3$.
3. $3x+6$ by $2x+3$.
4. $2x-5$ by $4x+7$.
5. $3x-2$ by $3x+3$.
6. $5-4a$ by $4a-7$.
7. $2x+y$ by $x+2y$.
8. $3m+4n$ by $3m-5n$.
9. $3x-2$ by $2x-3$.
10. $-3n+11p$ by $5n-p$.
11. $nx-px$ by $qx+rx$.
12. $-nx+p$ by $qx-nx^2$.

13. $3x - 2y^2$ by $5x + 4y$.

14. $n^2 - 5n + 6$ by $n - 3$.

15. $3p^2 - 3p - 7$ by $2p + 4$.

16. $a^2 - ab - b^2$ by $a + b$.

17. $b^2x^2 - abx + 4b^2$ by $bx + 2b$.

18. $x^3 - 2x^2 - 5x$ by $2x^3 - 4x^2$.

19. $2r^2 - 8r + 12$ by $r^2 - 3r - 4$.

20. $x^2 - 2xy^2 - 3y$ by $x^2 - 2xy - 3y^2$.

Expand:

21. $(2x^2 - 4x - 1)(2x - 2)$.

22. $(n^2 - 3n - 2)(n^2 - 2n + 3)$.

23. $(p^2 - 3p + 2)^2$.

24. $(n - n^2 - 3)^2$.

25. $(p^2 - 2p + 4)^2$.

26. $(3p - p^2 - 2)^2$.

27. $(4m - 2m^2 - 5)^2$.

28. $(3n^2 - 4n - 7)^2$.

29. $(3n^2 - 10np + 8p^2)^2$.

30. $(2b^2c^2 - 3bc + 1)^2$.

52. Powers. A *power* of a number is the product obtained by using the number as a factor one or more times.

For example, 4, or 2^2, is the second power of 2; 27, or 3^3, is the third power of 3; $64x^6$, or $(2x)^6$, is the sixth power of $2x$.

53. Arrangement. A polynomial is said to be *arranged* according to the *ascending* powers of a certain letter when the exponents of that letter in successive terms increase from left to right.

Thus, $8 + 3x - 5x^2 + 2x^4$ is arranged according to *ascending* powers of x.

Again, $y^2 - 3y + 4$ and $y^4 - 3xy^3 + 3xy^2 - x^3$ are arranged in *descending* powers of y.

Whenever it is possible, the multiplicand and the multiplier should be arranged in the same order with respect to some letter, as the addition of the partial products is then more easily performed.

54. Check of multiplication. The work of multiplication can be checked by giving a small numeric value to each letter involved and finding the corresponding numeric values of the multiplicand, the multiplier, and the product. The product of the numeric values of the first two should equal the numeric value of the product.

The smallest number which gives a reliable check is 2. This is true when only one letter is involved. If more letters are involved, the check is not certain if 2 is used.

The number 1 is sometimes convenient for checking, but it will not check exponents, since $a^2 = a^3 = a^4 = a^{15}$, etc., if $a = 1$.

EXAMPLE

Multiply $2x^3 - 7 - 4x + 3x^2$ by $x^2 - 6 - 5x$, and check.

Solution. Arranging both terms in descending powers of x and multiplying, we obtain:

If $x = 2$,

$$\begin{array}{rrrrrrrr}
2x^3 & +\ 3x^2 & -\ 4x & -\ 7 & & & = 16 + 12 - 8 - 7 = & 13 \\
x^2 & -\ 5x & -\ 6 & & & & = 4 - 10 - 6 \quad = & -\ 12 \\
\cline{1-4} \cline{8-8}
2x^5 & +\ 3x^4 & -\ 4x^3 & -\ 7x^2 & & & & -156 \\
 & -10x^4 & -15x^3 & +20x^2 & +35x & & & \\
 & & -12x^3 & -18x^2 & +24x & +42 & & \\
\cline{1-6}
2x^5 & -\ 7x^4 & -31x^3 & -\ 5x^2 & +59x & +42 & &
\end{array}$$

Check. $64 - 112 - 248 - 20 + 118 + 42 = -156$.

Since -156 is obtained in both parts of the check, the result is probably correct.

EXERCISES

Multiply and check:

1. $2x^2 - 3x - 8$ by $2x - 4$.
2. $3m^2 - 2m + 7$ by $2m^2 - 3m + 3$.
3. $2x^2 + x - 4$ by $x^2 + x + 2$.
4. $3n^2 - 8n - 1$ by $n^2 + 2n - 3$.
5. $3x^2 - 5x - 2$ by $3x^2 - 5x - 2$.
6. $3x^3 - 4x + 2$ by itself.
7. $h^2 - kh + k^2$ by $h^2 + kh + k^2$.
8. $n^2 - n + 1$ by $n^2 - 2n + 2$.
9. $3x^3 + 4x^2 - 2x + 3$ by $x^2 - 3x + 4$.
10. $s^3 - 2s^2 + s - 3$ by $3s^2 - 2s + 1$.
11. $3n - n^3 + 6$ by $4n - 3n^2 - 5$.

    HINT. Arrange both expressions in descending order.
12. $x^3 - x - 5$ by $3x^2 - 2x + 4$.
13. $3x - 2x^3 + x^2 - 8$ by $4 - x^2 - 4x$.
14. $3n^2 - 5n^3 + n + 1$ by $5 - n^2 + n$.

Expand:

15. $[5x - 2a - (2a - 5x)][5x - 2a + (2a - 5x)]$.
16. $(3a - 4a^2 + 6 + a^3)(2 + a^3 - 2a + 3a^2)$.
17. $(3x - 4 + 7x^3)(7 - 4x^2 + 2x^3 - 8x)$.
18. $(n^2 - 2np + 3p^2)(n^2 + 2np + 3p^2)$.
19. $(x^2y - y^2x)(3xy - 4x^2y)(2x^2y - xy^2)$.
20. $(n^2 + p^2 + q^2 - np - nq - pq)(n + p + q)$.
21. $(c + d + e)^2$.
22. $(2n - 3p + 4r)^2$.
23. $(3a - 5b + 6)^2$.
24. $(p - q - r)^2$.
25. $(n - 2p + 3r - 4s)^2$.

CHAPTER X

PARENTHESES IN EQUATIONS

55. Simple equations involving parentheses. In dealing with algebraic expressions involving parentheses great care must be exercised at all times. Accuracy in such work demands especial care in removing any parenthesis that is preceded by a minus sign.

EXAMPLE

Solve the equation $4(2x + 1) - (4x - 6) = 22$.

Solution. $4(2x + 1) - (4x - 6) = 22.$

Removing parentheses, $8x + 4 - 4x + 6 = 22.$

Combining, $4x + 10 = 22.$

Transposing, $4x = 12.$

Dividing by 4, $x = 3.$

Check. $4(2 \cdot 3 + 1) - (4 \cdot 3 - 6) = 22.$

Simplifying, $4 \cdot 7 - 6 = 22,$

or $22 = 22.$

EXERCISES

Solve and check:

1. $3(x + 3) = 15.$
2. $5(x - 1) = 25.$
3. $3(x + 6) = 12.$
4. $3(3 - x) + 2 = -1.$
5. $5(x - 4) + 14 = 9.$
6. $5(x - 8) + 7x = 8.$
7. $4(3x - 2) - 2 = 2x.$
8. $16 + 2(4n - 7) - 12n = 0.$
9. $5x - 3(5 - x) - 9 = 0.$
10. $2(x + 1) - 4 = 3(x - 1).$

11. $4(3x - 5) + 20 = 3(x + 9)$.
12. $3(x + 5) + 7 = 5(8 + x)$.
13. $8(x + 4) = 5(x + 8) + 4$.
14. $7(n - 3) - 2(3 - n) = 0$.
15. $9p - 3(2p - 4) = 2(5 - p) + 7$.
16. $3p - 9(3p + 4) = 3(p + 9)$.
17. $7p - 12 - 2(p - 6) = 2p - 15$.
18. $5(3n - 1) - 7n = 3(n + 6) - 3$.
19. $(p - 4)(p + 8) = 7 - (2 - p)(p + 3)$.

Solution. Expanding,

$$p^2 + 4p - 32 = 7 - (6 - p - p^2).$$

Removing parentheses,

$$p^2 + 4p - 32 = 7 - 6 + p + p^2.$$

Transposing and combining,

$$3p = 33.$$

Dividing by 3, $p = 11$.

Check. $(11 - 4)(11 + 8) = 7 - (2 - 11)(11 + 3)$.

Simplifying, $7 \cdot 19 = 7 - (-9 \cdot 14)$;

that is, $133 = 7 - (-126)$,

or $133 = 133$.

20. $(n + 2)^2 - (n + 3)^2 = -17$.
21. $(n + 2)^2 - (n - 3)^2 - 10 = 0$.
22. $(3x - 6)(4x - 8) = 12x^2 - 96$.
23. $(n + 3)(n + 5) = (n + 15)(n - 10)$.
24. $(p + 2)(p + 5) = (p - 4)(p - 7)$.
25. $(n - 7)(6 + n) - (n - 5)(n + 6) + 16 = 0$.
26. $(x - 5)(x + 3)(x + 2) - 5 = (x^2 - x - 1)(x + 1)$.
27. $(x^2 - 2x + 4)(x^2 + 2x + 4) - (x + 2) = (x^2 + 2)^2 + 20$.

ORAL EXERCISES

1. The length of a rectangle is $x + 2$ and its width is 3. What is its area? its perimeter?

2. The length of a rectangle is l and its width is w. What is its area? its perimeter?

3. A rectangle is $x + 2$ feet wide and $2x + 3$ feet long. What represents its area? its perimeter?

4. A rectangle is x feet wide and 3 feet longer than it is wide. What represents its length? its perimeter? its area?

5. One book costs a cents. What represents the cost of 3 books? 5 books? x books? $x + 2$ books? $x - 3$ books?

6. Represent the total area of two rectangles whose bases are $x + 2$ feet and $x + 3$ feet, and whose altitudes are $x - 3$ and $x + 5$ feet respectively.

7. Represent the area of a triangle whose base is x feet and whose altitude is 3 feet more than the base.

8. Represent the combined areas of two triangles, one of base $x + 2$ and altitude $x + 3$, and the other of base $2x - 1$ and altitude $2x + 3$.

9. What would represent the volume of a cube whose edge is x inches? whose edge is $x + 3$ inches?

10. A is x years old now. What will represent his age 2 years from now? 2 years ago? n years from now? n years ago?

11. A and B each have n dollars. If A gives B 10 dollars, how much will each have then? If A gives B x dollars?

12. A and B each have $x + 10$ dollars. B spends d dollars. A earns twice as much as B spends. What will represent the money they then have?

56. Problems involving parentheses. Two or more unknowns and the use of parentheses are involved in the solutions of the following problems. One unknown can always be represented by a single letter and the others by binomials involving this letter and one or more numbers. In some problems it will be necessary to inclose these binomials in parentheses and to think of them and use them as if each represented a single number.

EXAMPLE

The sum of two numbers is 47. Three times the less number is 4 less than twice the greater. What are the numbers?

Solution. Here 3 × less number = 2 × greater number − 4.

Let $n =$ the less number.

Then $47 - n =$ the greater number.

Substituting these symbols in the foregoing statement,

$$3n = 2(47 - n) - 4.$$

Removing parentheses,

$$3n = 94 - 2n - 4.$$

Transposing and combining,

$$5n = 90.$$

Dividing by 5, $n = 18.$

Check. $47 - n = 29.$

$$3 \times 18 = 2 \times 29 - 4.$$

Multiplying, $54 = 58 - 4.$

Combining, $54 = 54.$

PROBLEMS

1. The sum of two numbers is 53. Twice the greater equals seven times the less minus 11. What are the numbers?

2. The sum of two numbers is 30. Three times one of them minus twice the other equals 10. What are the numbers?

3. Find three consecutive numbers whose sum is 78.

4. Find three consecutive odd numbers whose sum is 87.

5. The sum of three numbers is 110. The first is 12 more than the second, and the third is 2 less than twice the second. What are the numbers?

6. The length of a hall is 12 feet more than three times the width. It requires 120 feet of base board for the hall. What are the dimensions of the hall?

7. Twice a number plus 4 equals three times the number less 19. What is the number?

8. A father is 2 years more than five times as old as his son. The sum of their ages is 38. How old is each?

9. The perimeter of a rectangle is 224 feet. It is two fifths as wide as it is long. What are its dimensions?

10. Twice a certain odd number plus three times the next consecutive odd number is 31. What are the numbers?

11. The altitude of a triangle is 2 inches shorter than its base. What are its dimensions if its area is 2 square inches more than half that of a square whose side equals the altitude of the triangle?

12. The sum of two numbers is 15. The difference of their squares is 15. What are the numbers?

13. The difference of the squares of two consecutive numbers is 31. What are the numbers?

14. The difference of the squares of two consecutive even numbers is 108. What are the numbers?

15. The product of two consecutive even numbers is 76 less than the square of the greater number. What are the numbers?

16. One number exceeds another number by 4, and its square exceeds the square of the second number by 48. Find the numbers.

Solution. (Greater number)^2 – (less number)$^2 = 48$.

Let $G =$ greater number.

Then $G - 4 =$ less number,

and $G^2 - (G - 4)^2 = 48,$

or $G^2 - (G^2 - 8\,G + 16) = 48.$

Removing parentheses and solving, we obtain $G = 8$.

17. Find two numbers such that their difference is 7, and the difference of their squares is 119.

18. Find two numbers whose sum is 18, and the difference of whose squares is 108.

19. Find three consecutive odd integers such that the product of the first and second is 154 less than the square of the third.

20. Find four consecutive odd numbers such that the product of the first and third is 72 less than the product of the second and fourth.

21. Find three numbers such that the second is 7 more than the first, the third 3 less than the first. The sum of the first and the square of the second is 103 more than the square of the third.

22. The difference between the squares of two consecutive odd numbers is 144. What are the numbers?

23. A room is 4 feet longer than it is wide. If the length is increased by 4 feet and the width decreased by 2 feet, the floor area will be increased by 8 square feet. How large is the room?

24. The length of a lot exceeds its width by 15 feet. If each were increased by 5, the area would be increased by 700 square feet. Find the dimensions of the lot.

25. Find a number such that if 6, 15, and 25 are in turn added to it, the product of the first and third sums will be 470 more than the product of the number and the second sum.

26. Divide $253 among eight men, four women, and five children, so that each woman shall receive three times as much as any child, and each man $6 more than any woman.

27. The value of 11 pieces of money, consisting of nickels and dimes, is 75 cents. How many of each are there?

HINT. Value of the nickels + value of the dimes = 75 cents.

Let $n =$ number of nickels.

Then $11 - n =$ number of dimes.

Now $5n =$ value of the nickels in cents,

and $10(11 - n) =$ value of the dimes in cents.

Therefore $5n + 10(11 - n) = 75.$

28. The value of 23 coins, consisting of nickels and dimes, is $1.60. How many of each are there?

29. The value of 37 coins, consisting of nickels, dimes, and quarters, is $4.25. There are twice as many dimes as nickels. How many coins of each kind are there?

30. A boy has 53 coins, consisting of nickels, dimes, and quarters, amounting to $5. He has twice as many nickels as dimes. How many of each kind has he?

31. There are 27 coins, consisting of nickels, dimes, quarters, and half-dollars, amounting to $5.95. There are 2 more dimes than nickels and as many half-dollars as nickels. How many of each kind are there?

32. A rectangle is the same width as a square, but is 2 feet longer. The difference between their areas is 32 square feet. What are the dimensions of each?

33. A certain rectangle is 5 feet longer than it is wide. If 2 feet are taken from the width and 3 feet from the length, the resulting rectangle is 24 square feet smaller than the original rectangle. Find the dimensions of the rectangle.

34. The radius of one circle is 2 feet more than that of another circle. The area of the first circle exceeds the area of the second circle by 88 square feet. Find the radius of each circle.

35. There are 18 steps in a certain stairway. The depth (tread) of each step is twice the height (rise). How deep is each step if 9 yards of carpet are required to carpet the stairs?

36. The radius of one circle is 2 feet less than the radius of a second circle. The difference of their areas is 22 square feet. What are the diameters of the two circles?

37. The side of one square is 7 feet less than the side of a second square. If two other squares be constructed, one with a side 2 feet greater than the side of the smaller square, and the other with a side 2 feet less than the side of the larger square, the difference of their areas will be 69 square feet. What are the dimensions of the original squares?

38. A is three times as old as B, and C is 8 years older than B. The sum of their ages is 48 years. How old is each?

39. A is 5 years older than B, and C is 7 years younger than B. Five years ago the sum of their ages was 43. How old is each?

HINTS. (A's age – 5) + (B's age – 5) + (C's age – 5) = 43.

Let $b =$ B's age in years now.

Then $b + 5 =$ A's age in years now,

and $b - 7 =$ C's age in years now.

Substituting these symbols, we have

$$b + 5 - 5 + b - 5 + b - 7 - 5 = 43.$$

Combining, $3\,b - 17 = 43.$

40. A is 5 years younger than B, and C is 10 years older than B. In 10 years the sum of their ages will be 110. How old is each now?

41. A is now 63 and B is 33. How many years ago was A three times as old as B?

42. A is now 38 and B is 59. How many years ago was A half as old as B?

43. The sum of the ages of A and B now is 30 years. In 9 years A's age will be three times B's age. How old is each?

44. A is four times as old as B. Five years ago he was three times as old as B will be one year from now. How old is each?

45. A grocer has 20 pounds of coffee worth 50 cents per pound. How many pounds of 35-cent coffee should he mix with it to produce a mixture worth 40 cents per pound?

46. A dealer has on hand walnuts priced at 45 cents a pound and Brazil nuts priced at 38 cents a pound. How many pounds of each should he take to make a mixture of 147 pounds which will be worth 39 cents per pound?

CHAPTER XI

DIVISION

57. Division by a monomial. Division is the process of finding one factor (the quotient) of a product of two factors (the dividend) when the other one of them (the divisor) is given.

Thus, the product 6 divided by the factor 3 gives the other factor, 2. Also $5\,x$ divided by 5 gives x, etc.

As in arithmetic, the indicated division may be written as a fraction and often simplified.

Thus, 2 divided by $5 = \frac{2}{5}$; and 9 divided by 12 is $\frac{9}{12} = \frac{3}{4}$. Similarly, $b \div c = \dfrac{b}{c}$; and $6\,a \div 8\,b$ is $\dfrac{6\,a}{8\,b} = \dfrac{3\,a}{4\,b}$.

By the definition of an exponent (p. 17),

$$a^3 = a \cdot a \cdot a \text{ and } a^5 = a \cdot a \cdot a \cdot a \cdot a.$$

Then $$\frac{a^5}{a^3} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = \frac{a^2}{1} = a^2 = a^{5-3}.$$

Similarly, $$\frac{3^5}{3^2} = \frac{\cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3 \cdot 3}{\cancel{3} \cdot \cancel{3}} = \frac{3^3}{1} = 3^3 = 3^{5-2}.$$

Also, $$\frac{nr^4}{r^2} = \frac{n \cdot \cancel{r} \cdot \cancel{r} \cdot r \cdot r}{\cancel{r} \cdot \cancel{r}} = nr^2 = nr^{4-2}.$$

Again, $$\frac{mn^3}{n^3} = \frac{m \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n}}{\cancel{n} \cdot \cancel{n} \cdot \cancel{n}} = m = mn^{3-3}.$$

These examples illustrate the

PRINCIPLE. *The exponent of any letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

This principle expressed in general terms is

$$n^a \div n^b = n^{a-b}.$$

What the above formula means when b is greater than a will be explained later when the need for such division arises. It will be seen then that this case is no exception to the principle expressed by the equation.

From what precedes we see that

$$\frac{m^5np^3}{m^3p^2} = m^{5-3}np^{3-2} = m^2np.$$

Also, $ax^2 \div x^2 = a.$

Hence a letter which has the same exponent in divisor and dividend should not appear in the quotient.

The law of signs in division (see Chapter III) may be indicated as follows:

$+bx$ divided by $(+b) = +x.$

$+bx$ divided by $(-b) = -x.$

$-bx$ divided by $(+b) = -x.$

$-bx$ divided by $(-b) = +x.$

Now $-16\,n$ divided by $8\,p = -\dfrac{2\,n}{p}.$

Here the quotient is a fraction and the minus sign indicates that the fraction is negative.

Similarly, $8\,x$ divided by $(-2\,y) = -\dfrac{4\,x}{y},$

and $-20\,n^2p$ divided by $(-5\,q^4) = +\dfrac{4\,n^2p}{q^4}.$

For the division of monomials we have the

Rule. *Divide the numeric coefficient in the dividend by the numeric coefficient in the divisor, following the rule of signs for division.*

Write after this quotient all the letters of the dividend except those having the same exponent in the dividend as in the divisor, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

If there are any letters in the divisor unlike those in the dividend, write these under the preceding result as a denominator.

ORAL EXERCISES

Divide:

1. -22 by 2.
2. 15 by (-3).
3. -32 by (-4).
4. $5a^3$ by a^2.
5. a^5 by a^2.
6. $-a^{12}$ by a^6.
7. x^5 by $(-x)$.
8. $-x^8$ by $(-x^2)$.
9. np^3 by $(-np)$.
10. $-n^2p^5$ by n^2p.
11. $-n^2xp$ by xp.
12. $n^2x^3p^4$ by n^2p.
13. $-9a^6$ by $3a^3$.
14. $8x^2$ by $(-2x)$.
15. $-24x^6$ by $(-6x^4)$.
16. $-35a^2x^3$ by $7a^2x$.
17. $15ax^3$ by $(-3bx^3)$.
18. $14a^3b$ by $(-7ab)$.
19. $-27n^3p^6$ by $(-3n^3p^2)$.
20. $-27np^4$ by $(-9np^3)$.
21. $77x^6y^9$ by $(-7x^2y^8)$.
22. $18a^2c^3x^2$ by $2ax^2c$.
23. $49ax^5$ by $(-7bx^4)$.
24. $-38x^6y^6$ by $(-2x^2y^3)$.
25. $-28n^6p^{12}$ by $(-4n^4p^5)$.
26. $-45n^4xp^8$ by $9n^3xp^7$.

27. $\dfrac{15n^2p^4}{75np^3}$.

28. $\dfrac{48n^{24}p^{56}}{-6n^6p^7}$.

29. $\dfrac{-17n^9p^8}{51n^7p^8}$.

30. $\dfrac{52n^{13}p^{26}}{-13n^{13}p^{13}}$.

31. $\dfrac{-11a^3b^{10}c^{12}}{77ab^2c^{10}}$.

32. $\dfrac{56n^7p^{14}r^{21}}{7n^2p^7r^7}$.

33. $\dfrac{-144a^6z^8}{-24a^4x^7}$.

34. $\dfrac{50x^3y^3z^3}{-2x^3yz}$.

35. $\dfrac{64mn^2p^3}{-16n^3p^3}$.

58. Division of a polynomial by a monomial. In multiplying a polynomial by a monomial (p. 92) we multiply each term of the polynomial by the monomial. In division the steps of multiplication are reversed and each term of the polynomial is divided by the monomial.

Thus, $(nx + px) \div x = \frac{nx}{x} + \frac{px}{x} = n + p.$

Again, $\frac{9\,x^4 - 12\,x^3 + 21\,x^2}{-3\,x^2} = -3\,x^2 + 4\,x - 7.$

Therefore, for the division of a polynomial by a monomial, we have the

Rule. *Divide each term of the polynomial by the monomial and write these results in succession.*

ORAL EXERCISES

Perform the indicated division:

1. $\frac{x^3 - x^4}{x^2}$.
2. $\frac{x^2y - x^3}{x^2}$.
3. $\frac{x^2 - x^4}{x^2}$.
4. $\frac{6\,x^2 - 4\,x}{2\,x}$.
5. $\frac{9\,x - 18\,x^3}{-3\,x}$.
6. $\frac{8\,a - 12\,a^3}{2\,a}$.
7. $\frac{8\,ax - 10\,a^2x^2}{-2\,ax}$.
8. $\frac{9\,ax^4 - 12\,a^3x^5}{-3\,ax^2}$.
9. $\frac{ab + ad}{a}$.
10. $\frac{25\,x^2y^2 + 30\,xy^4}{-5\,xy}$.
11. $\frac{16\,np^2 - 36\,n^2p^2}{4\,np^2}$.
12. $\frac{21\,x^3y^4 - 28\,x^5y^6}{7\,x^2y^3}$.

13. $\dfrac{8x^4y - 12x^6y^2 + 16x^8y^4}{4x^4y}$.

14. $\dfrac{rt^4 - st^3 + qt^2}{-t^2}$.

15. $\dfrac{12a^2b^2 + 9a^4b^3 - 27a^6b^4}{-3a^2b^2}$.

16. $\dfrac{16a^4b^5 - 24a^5b^6 - 32a^6b^7}{-8a^2b^3}$.

17. $\dfrac{75xyz - 45x^2yz^2 + 90x^3yz^3}{15xyz}$.

18. $\dfrac{5(x+3) + b(x+3)}{x+3}$.

19. $\dfrac{3(x+1) - 2x(x+1)}{x+1}$.

20. $\dfrac{(a+b) - 2(a+b)^2}{a+b}$.

21. $\dfrac{3a(3x+4) - 4a(3x+4)}{3x+4}$.

22. $\dfrac{(a-b)c - (a-b)d}{a-b}$.

23. $\dfrac{(x-y) - 3(x-y)}{-2(x-y)}$.

24. $\dfrac{16(3x-4)^4 - 24(3x-4)^5 - 48(3x-4)^7}{-8(3x-4)^4}$.

25. $\dfrac{-5(ac^2-2d)^3 + x(ac^2-2d)}{5(ac^2-2d)}$.

26. $\dfrac{(a+b)(x-y) + 7(x-y)}{x-y}$.

59. Division of one polynomial by another. In arithmetic the work of long division is commonly arranged as follows:

$$\begin{array}{rll}
 & 2276 & \text{Quotient}\\ \cline{2-2}
\text{Divisor, } \underline{346} & \big)\,787496 & \text{Dividend}\\
 & \underline{692} & \\
 & 954 & \\
 & \underline{692} & \\
 & 2629 & \\
 & \underline{2422} & \\
 & 2076 & \\
 & \underline{2076} &
\end{array}
\qquad\qquad
\begin{array}{lr}
\textit{Check.} & 2276\\
 & \underline{346}\\
 & 13656\\
 & 9104\\
 & \underline{6828}\\
 & 787496
\end{array}$$

It should be noted that in the above

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient},$$

and $$\text{Quotient} \times \text{Divisor} = \text{Dividend}.$$

These relations hold in algebra as in arithmetic.

In algebra, division by a polynomial is carried out in a similar manner, as is seen in the following

EXAMPLES

1. Divide $2x^2 - 10x + 12$ by $x - 2$.

Solution *Check*

$$\begin{array}{rll}
 & 2x - 6 & \text{Quotient}\\ \cline{2-2}
\text{Divisor, } \underline{x-2} & \big)\,2x^2 - 10x + 12 & \text{Dividend}\\
 & \underline{2x^2 - 4x\phantom{{}+12}} & = (x-2) \cdot 2x.\\
 & - 6x + 12 & \\
 & \underline{- 6x + 12} & = (x-2) \cdot -6.
\end{array}
\qquad
\begin{array}{r}
2x - 6\\
\underline{x - 2}\\
2x^2 - 6x\phantom{{}+12}\\
\underline{-4x + 12}\\
2x^2 - 10x + 12
\end{array}$$

2. Divide $16a + 12a^3 - 15 - 22a^2$ by $2a - 3$.

Solution

$$
\begin{array}{r|l}
 & 6a^2 - 2a + 5 \\ \hline
2a - 3 & 12a^3 - 22a^2 + 16a - 15 \\
 & \underline{12a^3 - 18a^2} \\
 & \qquad -4a^2 + 16a \\
 & \qquad \underline{-4a^2 + 6a} \\
 & \qquad\qquad +10a - 15 \\
 & \qquad\qquad \underline{+10a - 15}
\end{array}
$$

Check

$$
\begin{array}{rrrr}
6a^2 & - 2a & + 5 & \\
2a & - 3 & & \\ \hline
12a^3 & - 4a^2 & + 10a & \\
 & - 18a^2 & + 6a & - 15 \\ \hline
12a^3 & - 22a^2 & + 16a & - 15
\end{array}
$$

The method of dividing one polynomial by another is expressed in the

Rule. *Arrange both the dividend and the divisor according to the descending (or ascending) powers of some common letter, called the letter of arrangement.*

Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.

Multiply the divisor by the first term of the quotient and subtract, being careful to write the terms of the remainder in the same order as those of the divisor.

To find the second term of the quotient, divide the first term of the remainder by the first term of the divisor, and proceed as before until there is no remainder, or until the remainder is of lower degree in the letter of arrangement than the divisor.

Check. *Quotient × Divisor = Dividend.*

60. Numeric check for division. The following numeric check for division is often useful.

Rule. *Assign a small numeric value to the letter or letters of the dividend, divisor, and quotient. Then divide the numeric value of the dividend by that of the divisor. The result should equal the numeric value of the quotient.*

This check has certain advantages over the check stated in the general rule for the division of polynomials, as it is shorter and involves a totally different series of operations from those used in the work of the division itself. It should be noted, however, that while 1 is the easiest number to use, it does not check errors in exponents. Hence 2 or 3 are better numbers. *Any number which makes the divisor zero must never be used.*

This method of checking division is illustrated for Example 1 (p. 113), as follows:

Check. Let x have a numeric value of 4.

Then $2x^2 - 10x + 12$ will have a value of 4 (Dividend),

$x - 2$ will have a value of 2 (Divisor),

and $2x - 6$ will have a value of 2 (Quotient).

Here $4 \div 2 = 2.$

It will be noted in the example cited that the value $x = 2$ would have made the divisor equal to zero. The value $x = 3$ could have been used, but it gave a value of the dividend equal to zero, and was avoided for the sake of clearness.

EXERCISES

Divide the following and check as directed by the teacher:

1. $a^2 + 10a + 24$ by $a + 4$.
2. $n^2 + n - 6$ by $n - 2$.
3. $5n^2 + 7n + 2$ by $n + 1$.
4. $2p^2 - 3p - 2$ by $2p + 1$.
5. $n^2 - 2n - 15$ by $n + 3$.
6. $5n^2 - 22n + 8$ by $n - 4$.
7. $3x^2 + 8x - 3$ by $x + 3$.
8. $6n^2 + 19n - 7$ by $3n - 1$.
9. $3p^2 - ap - 2a^2$ by $p - a$.
10. $n^3 - 11n - 6$ by $n + 3$.

11. $4p^2 - 8np + 3n^2$ by $2p - 3n$.
12. $8n^3 - 12n^2 + 6n - 1$ by $2n - 1$.
13. $a^3 - 14a - 8$ by $a - 4$.
14. $n^3 - 11n + 6$ by $n^2 + 3n - 2$.
15. $p^3 - 15p^2 + 65p - 63$ by $p - 7$.
16. $p^3 - 11p + 6$ by $p - 3$.
17. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
18. $2n^3 - 14n^2 + 14n + 12$ by $2n - 4$.
19. $3n^3 + 28n^2 + 89n - 240$ by $3n - 5$.
20. $37p + 6p^3 - 24 - 23p^2$ by $2p - 3$.
21. $-40p - 31p^2 + 21 + p^4 + 4p^3$ by $p^2 - 3 + 7p$.
22. $n^3 + 8$ by $n + 2$.
23. $8p^3 + 1$ by $2p + 1$.
24. $8n^3 - 125p^3$ by $2n - 5p$.
25. $n^3 + 125p^3$ by $n + 5p$.
26. $27n^3 + 8p^3$ by $3n + 2p$.
27. $n^3 - p^3$ by $n - p$.
28. $n^3 + p^3$ by $n + p$.
29. $n^6 + 343p^3$ by $n^2 + 7p$.
30. $x^4 - 16$ by $x + 2$.
31. $x^4 - 16$ by $x - 2$.

61. Inexact division. In division it frequently happens that the divisor is not contained in the dividend exactly; that is, without a remainder.

Thus, 25 divided by 4 gives a quotient of 6 and a remainder of 1. This is expressed in

$$\tfrac{25}{4} = 6 + \tfrac{1}{4}, \text{ or } 6\tfrac{1}{4}.$$

Similarly, $$\frac{17a}{5} = 3a + \frac{2a}{5}.$$

The result of division when there is a remainder can be expressed in general terms thus:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

EXAMPLE

Divide $6a^2 - 13a + 3$ by $3a - 2$.

Solution.

$$\begin{array}{r|l} & 2a - 3 - \frac{3}{3a-2} \\ \hline 3a-2 & 6a^2 - 13a + 3 \\ & 6a^2 - 4a \\ \hline & \quad -9a + 3 \\ & \quad -9a + 6 \\ \hline & \qquad\quad -3 \end{array}$$

Check. $(2a - 3) \times (3a - 2) - 3 = 6a^2 - 13a + 6 - 3.$

$$= 6a^2 - 13a + 3.$$

62. General check for division. (*a*) *When the division is exact, multiply the divisor by the quotient. The product should be the dividend.*

(*b*) *When there is a remainder, multiply the divisor by the partial quotient and add the remainder to the product obtained. The result should be the dividend.*

NOTE. We saw on page 2 that it is customary to represent the product of two letters by placing one after the other with no sign between them. Thus, ab means a times b. But addition, not multiplication, is implied by placing the fraction $\frac{2}{5}$ after the number 3. This practice comes down to us from the Arabs, who denoted all additions by placing the number symbols in succession without any sign of operation. The later Greeks also had the same notation.

EXERCISES

Divide the following and check as directed by the teacher:

1. $6n^2 - 13n - 6$ by $2n - 3$.
2. $19p + 12p^2 + 21$ by $4p - 3$.
3. $6n^2 + 11n - 35$ by $2n - 7$.
4. $25p^4 + 30p^2 - 7$ by $5p^2 - 7$.

5. $-2n^2 - 8 + n^3 + 4n$ by $n - 2$.
6. $n^3 - 15n^2 + 65n - 66$ by $n - 6$.
7. $p^3 - 3p^2c + 3pc^2 - c^3$ by $p - c$.
8. $10x^2 + 10x - 50x^3 - 1$ by $5x^2 - 1$.
9. $6p^3 - 18p + 12$ by $p - 1$.
10. $3p^3 - p^2 + 5p + 9$ by $p^2 - 2p + 9$.
11. $a^3 + 28a^2 - 18a - 130$ by $a - 3$.
12. $23x^2 - 37x - 24 - 6x^3$ by $2x + 3$.
13. $n^2 + 5np - 2n + 10p$ by $n + 5p$.
14. $4n^2 + 3cn + 8an + 6ac$ by $4n + 3c$.
15. $53p + 8 - 53p^2 + 12p^3$ by $4p^2 - 7p - 1$.
16. $25p^3 - 10p^2 + 36p + 72$ by $5p - 6$.
17. $2an + 5xy - 5ny - 3ax$ by $3x - 2n$.
18. $3n^4 + 11n^3 - 3n^2 + 17n - 4$ by $n - 3n^2 - 4$.
19. $23n^2 + n^4 - 55n + 11n^3 - 140$ by $n^2 - 5$.
20. $4p^3 + 1 + p^4 + 4p + 6p^2$ by $2p + p^2 + 1$.
21. $21 + 40p - 31p^2 + p^4 - 4p^3$ by $p^2 - 3 - 7p$.
22. $n^4 - 8n^3 + 24n^2 - 32n + 16$ by $n^2 - 4n + 4$.
23. $224n - 42n^2 + 10n^4 - 27n^3 - 419$ by $9 + 2n^2 - 5n$.
24. $x^5 - 5x^4 + 9x^3 - 6x^2 - x + 2$ by $x^2 - 3x + 2$.
25. $30x^4 + 11x^3 - 82x^2 - 12x + 48$ by $2x - 4 + 3x^2$.
26. $138y - 36 - 142y^3 + 56y^4 - 70y^2$ by $4y^2 - 13y + 6$.
27. $12x^5 - 30x^4 + 8x^3 + 14x^2 - 14x + 4$ by $6x^3 - 2x + 2$.
28. $4y^3 - 16y + 2y^4 + 24 - 14y^2$ by $y^2 + 2 - 6y$.
29. $28x^4 - 90x^3y + 156x^2y^2 + 90xy^3 + 28y^4$ by $4x^2 + 10xy$.

30. $n^3 - p^3$ by $n - p$.
31. $n^6 - 1$ by $n - 1$.
32. $n^6 - 1$ by $n + 1$.
33. $n^5 + y^5$ by $n + y$.
34. $n^5 + p^5$ by $n - p$.
35. $n^5 - p^5$ by $n + p$.

36. $n^5 - p^5$ by $n - p$.

37. $p^5 - 5p^2 - 3000$ by $p - 5$.

38. $64n^6 + 27p^{12}$ by $4n^2 + 3p^4$.

39. $p^2 - 2np + n^2 - a^2$ by $p - n + a$.

40. $81p^4 - 256$ by $27p^3 + 36p^2 + 48p + 64$.

41. $6n^4 - 7n^3 - 28n^2 + 8n - 21$ by $3n^2 - 5n - 7$.

42. $p^4 + p^2n^2 + n^4$ by $p^2 + pn + n^2$.

43. $8p^4 - 50p^2 + 32$ by $4p^2 + 6p - 8$.

44. $n^5 - n^4p - np^4 + p^5$ by $n^2 - 2np + p^2$.

45. $n^3 - 3npr + p^3 + r^3$ by $n + p + r$.

REVIEW EXERCISES

Name the factors in:

1. $4a^2xyz^3$. 2. $7a^3bc^5d^3$. 3. $10x^5y^{10}z$. 4. $9x^{10}yz^5$.

What is the coefficient of x in:

5. $9abx^7$. 6. $3ay^2x$. 7. $15aw^5x^4$. 8. $10abcx^2yz$.

9. What is the coefficient of a in Exercises 5–8?

10. What is the coefficient of b in Exercises 5–8? of y? of z?

What is the exponent of x in:

11. $7ax^5$. 12. $6xy^2$. 13. $9x^3m$. 14. xac^2.

When $x = 2$, $y = 3$, $z = 5$, find the value of:

15. $\sqrt{z^2 - y^2}$. 17. $(x + y + z)^2$. 19. $x^2 + z^3$.

16. $\sqrt[3]{y + z}$. 18. $(x + y)^2 - (z - y)^2$. 20. $\sqrt{yz + xz}$.

21. $\sqrt{z^2 + y^2 + x}$. 22. $\sqrt{3yz + x^2}$.

23. If $x = 2$ and $y = -3$, find the value of $3\,xy^2$; $5\,xy$; $2\,x^2y^3$; $7\,xy - 4\,x^2y$.

24. From the sum of $a^2x + ax^2 + 3\,ay$, $4\,a^2x - 2\,ax^2 + 5\,ay$, $7\,ax^2 - 6\,ay$, and $-4\,a^2x - 3\,ax^2 + 3\,ay$ take the sum of $5\,a^2x - 3\,ax^2 - 5\,ay$, $-12\,a^2x + 2\,ax^2 + 7\,ay$, and $7\,ax^2 - 3\,ay$.

Simplify:

25. $2\sqrt{x-y} - 3\sqrt{x-y} + 7\sqrt{x-y} - 6\sqrt{x-y}$.

26. $a(a+b) + b(a+b) - c(a+b) - 3(a+b)$.

27. $6(x-2\,y) + 3(x-2\,y) - 5(x-2\,y) - a(x-2\,y)$.

28. $2\,ab(w+v) + 3\,ab(w+v) - 7(w+v) - ab(w+v)$.

29. $(x-y)\sqrt{ab} + 12\sqrt{ab} - 4\sqrt{ab} - (x-y)\sqrt{ab}$.

If $A = a^2 + 4\,ab - 3\,b^2$, $B = 2\,ab + 4\,b^2 - 3\,a^2$, $C = b^2 + 2\,a^2 - 3\,ab$, $D = a^2 - 2\,b^2 + 2\,c^2$, find the expression for:

30. $A + B - C - D$.

31. $A - B + C - D$.

32. $-2\,A + B + C - 2\,D$.

Find the values of the unknown in:

33. $2\,m + 5 = m + 7$.

34. $6(n+3) = 4(n+6)$.

35. $\dfrac{n+5}{2} = 3\,n - 10$.

36. $2(b-3) = b + 1$.

37. $2\,x + 5 + x = 4\,x - 2$.

38. $3(x+2) = 4(x - \frac{1}{2})$.

Which of the following equations are identities and which are equations of condition?

39. $3\,a - 7 = 2\,a$.

40. $x - 3 = 3\,x - 4$.

41. $3(x+1) = 3\,x + 3$.

42. $3(x-3) = 5\,x + x - 4 + x + 1 - 2(x+6)$.

Remove the parentheses and collect like terms:

43. $3x - 4y - [3z - 8x - (2y - 3z) - 9x + 2z]$.

44. $4a - [3a - (8a + 2b) + 4] - (3b - 6)$.

45. $[4x + (3z - 4y)] + [4x - (3z - 4y)]$.

Multiply and check:

46. $a - 2 - a^3 + 2a^2$ by $2a - a^2 - 1$.

47. $(x^2y - 2y^2x)(xy - 5x^2y)(3x^2y - xy^2)$.

48. $(a - 3b + 2c)^2$.

49. $(a - b + 2c - 3d)^2$.

50. $(2x - 4y)^2 - (2x + 3y)^2$.

51. $[a + (2a^2 + 3)][a - (2a^2 + 3)]$.

Solve for the unknown:

52. $(k - 7)(4 + k) - (k - 5)(k + 8) = 0$.

53. $(3x - 5)(4x - 2) = 12x^2 - 172$.

54. $(b - 5)(2b + 3) + 21 = 2(b - 5)(b + 5)$.

Divide:

55. $\dfrac{4(a^2c - 2d)^2 + x(a^2c - 2d) - 2(a^2c - 2d)^3}{2(a^2c - 2d)}$.

56. $\dfrac{x^5 + 5x^4y + 11x^3y^2 + 4x^2y^3 - 2xy^4 + y^5}{x + y}$.

57. A woman bequeathed \$10,500 to three charities. To the second she bequeathed two fifths more than to the first, and to the third \$60 less than twice the amount given to the first. How much did each receive?

58. A man and a boy together earn \$55. The man receives three times as much per day as the boy. How much does each receive if the boy works 10 days and the man works 4 days?

59. A certain number is reduced by one half and then by three eighths of its original value. The result is 125. Find the number.

60. There are 112 bushels of corn in 2 bins. In one bin there are 17 bushels less than half as many as there are in the other. How many bushels are there in each?

61. A boy bought oranges at the rate of 3 for 2 cents and gained 50 cents by selling them 2 for 3 cents. How many oranges did he buy?

62. A cow and a sheep together cost $125. The cow cost $5 more than three times as much as the sheep. Find the cost of each.

63. An estate of $97,800 was left to three men. The second received $1200 more than the first, and the third $600 less than the first and the second together. How much did each receive?

64. Three men have together $2000. The first has $275 more than the second, and the second has $300 more than the third. How much has each?

65. Three men receive a certain sum of money. The first receives twice as much as the second, the second twice as much as the third. The difference between the amounts received by the first and third is $7500. How much does each receive?

66. A stick 117 inches long is to be cut so that the first piece is 2 inches more than twice as long as the second, and the second 5 inches less than three times as long as the third. Find the length of each piece.

67. Four boys together sold 100 papers. The first sold twice as many as the second, the third 5 more than three times as many as the second, and the fourth 4 less than the first and second together. How many papers did each sell?

68. Find three consecutive numbers such that the product of the first and third exceeds the product of the first and second by 27.

69. Find three consecutive numbers whose sum is 84.

70. Find three consecutive even numbers such that the product of the second and third less the product of the first and third is 16 less than three times the first.

71. Find three consecutive odd numbers such that the product of the second and third exceeds the product of the first and second by 220.

CHAPTER XII

EQUATIONS AND PROBLEMS

63. Equations involving literal coefficients. In the most general form of a simple equation in one unknown the constant term is represented by a letter and the unknown has literal coefficients. The simplest general form is $ax = b$, in which x is the unknown and a and b are known numbers. The solution of such equations involves no new principle. It is merely necessary to perform the usual operations with letters instead of with arithmetic numbers.

In solving equations whose coefficients involve letters the answers obtained will usually involve these same letters. Only in exceptional cases may one expect to obtain the root of such an equation in purely numeric form.

In the following exercises the unknown is represented by a letter near the end of the alphabet, as x or y, and the letters in the coefficients and terms supposed to be known are taken from the other letters of the alphabet.

ORAL EXERCISES

Solve for x, y, v, or z:

1. $x - c = 0$. 4. $x - 7m = 2m$. 7. $az + ab = ac$.
2. $x + 3b = 8b$. 5. $x - 3m = 1 - m$. 8. $nz - an = bn$.
3. $x + b = c$. 6. $3z - 6b = 9m$. 9. $v - 6a = 12a$.

10. $5v = 10m$.
11. $2av = 6a^2$.
12. $mv - 3m^2 = m^2$.
13. $av = 4a^3$.
14. $ay = ab + a^2$.
15. $2y = 4m - n$.
16. $3cy = 9c - 6c^3$.
17. $4ay - 2a^2 = 6a^2$.
18. $3cy + c^2 = 7c^2$.
19. $(m + n)x = m + n$.
20. $(m - n)y = (m - n)^2$.
21. $(m + n)z = 2a(m + n)$.
22. $(n + t)(n - t)x = (n + t)(n - t)$.
23. $(n - t)(n + t)y = 2c(n + t)(n - t)$.

EXERCISES

Solve for the unknown, and check:

1. $5x - 2a = 10a + 3x$.

Solution. $5x - 2a = 10a + 3x$.
Transposing, $5x - 3x = 10a + 2a$.
Combining, $2x = 12a$.
Dividing by 2, $x = 6a$.

Check. Substituting $6a$ for x in the first equation,

$$5(6a) - 2a = 10a + 3(6a)$$
$$30a - 2a = 10a + 18a, \quad \text{or} \quad 28a = 28a.$$

2. $mx + m = 5m$.
3. $x - m = m + n$.
4. $x - c = m - c$.
5. $3mx - 5m = 7m$.
6. $ay - am = ac$.
7. $my - m^2 = 5m^2$.
8. $3bx - b^2 = 8b^2$.
9. $mn + nx = 5mn$.
10. $5ax - 10a^2 = 5ac$.
11. $am - mx = 3am$.
12. $3(a + x) = 6a$.
13. $4(a - x) = 10a$.
14. $12c - 3(c - x) = 0$.
15. $ay - (a - c) = 3a + c$.
16. $3ay - ab = 2ay - ac$.
17. $4nz - 7n^2m = 3nz + 6mn^2$.

18. $3\,ax + 2\,ab = 6\,ab + 2\,ax - 3\,ab.$

19. $mx + n^2 = 4\,m^2 - (mx - n^2).$

20. $ax + 5\,a = a^2 + 6 + 3\,x.$

Solution. $ax + 5\,a = a^2 + 6 + 3\,x.$

Transposing, $ax - 3\,x = a^2 - 5\,a + 6.$

Uniting the coefficients of x,

$$(a - 3)x = a^2 - 5\,a + 6.$$

Dividing both members by $a - 3$,

$$x = \frac{a^2 - 5\,a + 6}{a - 3}.$$

Performing the division,

$$x = a - 2.$$

Check. Substituting $a - 2$ for x in the original equation,

$$a(a - 2) + 5\,a = a^2 + 6 + 3(a - 2).$$

Simplifying, $a^2 - 2\,a + 5\,a = a^2 + 6 + 3\,a - 6,$

or $a^2 + 3\,a = a^2 + 3\,a.$

21. $ax + bc = bx + ac.$

22. $my + 1 - m^2 - y = 0.$

23. $ay + 2\,ab = 2\,a^2 + by.$

24. $nx + 1 - n^3 - x = 0.$

25. $ax - 2\,a^3 - 1 = a^2 - x.$

26. $x - 5\,n^2 = 2\,n^3 - 2\,nx - 1.$

27. $6\,mc + ax + an = 3\,am + 2\,cx + 2\,cn.$

28. $2\,a(a + c) + 3\,nx = a(3\,n + 2\,x) + 3\,nc.$

HINT. Remove parentheses first.

29. $(y - a)(y + n) + an + a^2 = y^2 + n^2.$

30. $15(y - n) - 6(y + n) = 3(5\,n - 3\,y).$

31. $4\,x - cx - 8 - 4\,a + 6\,c = 2\,cx - 3\,ac.$

32. $2\,ny - 2\,n = 1 - y.$

64. Uniform motion. If a man rides for 8 hours at the uniform (unvarying) speed of 20 miles per hour, he will travel in all 8 × 20, or 160, miles. This example illustrates *uniform motion.* In all problems of uniform motion the elements involved are

(*a*) Time, measured in seconds, minutes, hours, etc.

(*b*) Rate of motion (speed); or the distance traveled in a unit of time (one second, one hour, one day, as the case may be).

(*c*) Distance (total), measured in feet, inches, miles, meters, etc.

If a body moves at a constant speed r for a time t through a distance d, then d, r, and t are connected by the equation

$$d = r \times t. \qquad (1)$$

This relation or formula gives an insight into the power of the algebraic language. By means of arithmetic numbers alone we can express the relation which holds between the time, the speed, and the distance only for a particular case. By means of the literal equation (1) we express the relation which is true not merely for one case but for countless cases. In fact, it holds whenever we are dealing with uniform motion, whatever numeric values d, r, and t may have.

ORAL EXERCISES

1. An automobile ran 35 miles per hour for 7 hours. How far did it run?

2. An automobile runs 320 miles in 8 hours. Find the average speed.

3. A steamship travels 1920 miles in 4 days. What is its speed in miles per hour?

4. How far does an automobile travel if it runs

(*a*) 25 miles per hour for 6 hours?
(*b*) 20 miles per hour for h hours?
(*c*) m miles per hour for h hours?
(*d*) 30 miles per hour for $t + 5$ hours?
(*e*) $2x - 4$ miles per hour for 7 hours?

Sound travels about 1100 feet per second. Light travels 186,300 miles per second. Therefore the interval of time between the occurrence and the observation of an event may be ignored in the following problems:

5. An observer hears the stroke of an ax $1\frac{2}{5}$ seconds after he sees the ax strike the tree. How far is the observer from the woodcutter?

6. How far is an observer from a rocket if he hears it $7\frac{1}{2}$ seconds after he sees the flash of the explosion?

7. What is the speed of an automobile if it runs uniformly

(*a*) 240 miles in 8 hours?
(*b*) d miles in 8 hours?
(*c*) d miles in h hours?
(*d*) 240 miles in $x + 3$ hours?
(*e*) d miles in $t - 2$ hours?

8. The United States Air Mail Service has a schedule of 32 hours from New York to San Francisco, a distance of 2620 miles. What is the average speed in miles per hour?

9. An automobile runs a distance d at a speed of 25 miles per hour. Another car runs 90 miles farther at 30 miles per hour. Represent the number of hours required by each car for its run. If each requires the same time, what equation can be stated?

10. An automobile runs 25 miles per hour for t hours, a second one runs 18 miles per hour for $t + 3$ hours. Represent the distance each travels. If the two cars travel the same distance, what equation can be stated?

11. If the first automobile in Exercise 10 runs 20 miles farther than the second, what equations can be stated?

12. Two automobiles leave two towns 450 miles apart at the same time and travel toward each other, one at the rate of t miles per hour, the other twice as fast. They meet in 10 hours. Represent the distance each has traveled during that time. What equation involving t exists?

13. An aëroplane leaves one city to fly to another 720 miles away. An hour later another aëroplane leaves the second city to fly to the first. The two meet midway between the cities. Represent the speed of each. If one travels 30 miles per hour faster than the other, state the equation involving t which represents the conditions of the problem.

EXAMPLES

1. A motorist traveling 20 miles per hour is overtaken in 12 hours after leaving a certain point by a second motorist, who left the same starting point 4 hours after the first. Find the speed of the second motorist.

Solution. This is a problem in uniform motion, involving the distance, the speed, and the time of the first and of the second motorist. By a careful reading of the problem one discovers that the time of each was a different number of hours, that each went at a different speed, but that each traveled the same distance. Hence an equation can be formed by expressing d in terms of r and t for each and then equating the two expressions for d.

By the conditions of the problem:

	t, or Time in Hours	r, or Rate in Miles per Hour	Distance in Miles $r \times t = d$
First motorist	12	20	$20 \times 12 = 240$
Second motorist	8	x	$8x$

and $8x = 240,$

then $x = 30.$

Check. $20 \times 12 = 240.$ $30 \times 8 = 240.$

2. Two men, A and B, start from the same place at the same time and travel in opposite directions. B goes three times as fast as A. In 6 hours they are 96 miles apart. Find the speed of each.

Solution. By the conditions of the problem:

	t, or Time in Hours	r, or Rate in Miles per Hour	Distance in Miles $r \times t = d$
A	6	r	$6r$
B	6	$3r$	$18r$

and $6r + 18r = 96;$

then $24r = 96,$

whence $r = 4$ and $3r = 12.$

Check. $12 = 3 \cdot 4;\ 6 \cdot 4 + 6 \cdot 12 = 96.$

It should be particularly noted in choosing letters for the unknowns that it is not enough to say, for instance, let x equal the distance or let t equal the time. This means little unless the unit of distance or the unit of time is stated. The unknown distance is a number of feet or miles or some other unit of length, and the unknown time is a

Sir Isaac Newton

number of seconds or hours or some other specific unit of time. This should be borne in mind each time a letter is taken to represent the measure of any physical quantity.

BIOGRAPHICAL NOTE

Sir Isaac Newton. Sir Isaac Newton (1642–1727) was probably the keenest mathematical thinker who ever lived. He was the son of a farmer of slender means, and as a boy was rather lazy. It is said, however, that his complete victory over a larger boy in a fight at school led him to feel that perhaps he could be equally successful in his studies if he really tried. His ambition and interest being once aroused, he never ceased to apply himself during the rest of his long life.

His most important scientific achievement was the discovery and verification of the laws of motion. In his great work called the "Principia" he showed by mathematical reasoning that all bodies, great and small, — the planet revolving around the sun, as well as the apple falling from the tree, — follow the same laws. His greatest discovery in pure mathematics was that of a method called the calculus, which is the basis of most of the advances in mathematics and in theoretical physics made since his time.

But important as was Newton's mathematical work, his most significant contribution to mankind was an idea, — the idea that the world in which we live is not independent of the rest of the universe, but that every smallest particle of matter is connected with the most remote planet and star; that we cannot think of the earth as the center of all things, but that we merely occupy our place in a system governed by universal law.

MISCELLANEOUS PROBLEMS

In Problems 1–5, A and B start at the same time from two points 168 miles apart and travel toward each other. Find the speed of each:

1. If they travel at the same speed and meet in 7 hours.

2. If A travels 2 miles per hour faster than B and they meet in 12 hours.

3. If B travels three times as fast as A and they meet in 7 hours.

4. If A travels 48 miles farther than B and they meet in 6 hours.

5. If they meet in 14 hours and A travels 2 miles per hour less than B.

6. C and D travel the same distance, C in 8 hours, D in 10. C's speed is 3 miles per hour more than D's. Find the speed of each and the distance traveled.

7. The first transatlantic non-stop flight was made by Alcock and Brown from St. John's, Newfoundland, to Clifden, Ireland, in June, 1919. Their speed was $121\frac{1}{9}$ miles per hour. Had it been $21\frac{1}{9}$ miles per hour less, the time taken would have been increased 3.42 hours. Find the distance flown.

In Problems 8–11, A and B start at the same time from two points 252 miles apart and travel toward each other until they meet. Find the number of hours from the start until the time of meeting:

8. If A travels 6 miles per hour faster than B and twice as far.

9. If A travels 5 miles more per hour than B and also travels three times as far.

10. If B travels 9 miles per hour and A travels 12 miles per hour but is delayed 2 hours on the way.

11. If B is delayed 2 hours and A is delayed 6 hours and their rates while moving are 15 and 17 miles per hour respectively.

12. A starts from a certain place and travels 20 miles per hour. Two hours later B starts from the same place and travels in the same direction at the rate of 25 miles per hour. In how many hours will B overtake A?

13. On May 2, 1923, Lieutenants Kelly and Macready of the United States Air Service with the plane T-2 broke all previous records by flying from New York to San Diego in 26 hours and 51 minutes. Had they flown 167 miles farther in the same time their rate would have been 100 miles per hour. Find their average speed and the distance from New York to San Diego.

14. A traveler having 4 hours at his disposal wishes to ride out of town on a trolley car whose speed is 12 miles per hour and return on foot at a rate of 3 miles per hour. On the way back his route is 2 miles longer than that of the car going out. How long and how far may he ride?

The velocity of a bullet continually decreases from the instant it leaves the gun. This is owing to the resistance of the air. In the following problems consider the velocity of sound to be 1100 feet per second:

15. A marksman hears the thud of a bullet striking a target 880 yards away 4 seconds after he presses the trigger. Find the average velocity of the bullet.

16. Two seconds after a marksman presses the trigger of his revolver he hears the bullet strike the target 60 rods away. Find the average velocity of the bullet.

17. The distance from New York to Chicago is 912 miles. The Manhattan Limited makes the run in 22 hours. An eastbound train leaves Chicago at the same time the Manhattan Limited leaves New York. The two pass each other twelve hours later. Find the speed of each train.

CHAPTER XIII

IMPORTANT SPECIAL PRODUCTS

65. The square of a binomial. The multiplication

$$\begin{array}{rrr}
a & +b & \\
a & +b & \\
\hline
a^2 & +ab & \\
 & +ab & +b^2 \\
\hline
a^2 & +2ab & +b^2
\end{array}$$

gives the formula $(a+b)^2 = a^2 + 2ab + b^2$.

This may be expressed in words as follows:

I. *The square of the sum of two terms is the square of the first term plus twice the product of the two terms plus the square of the second term.*

Similarly, $(a-b)^2 = a^2 - 2ab + b^2$,

which may be expressed in words as follows:

II. *The square of the difference of two terms is the square of the first term minus twice the product of the two terms plus the square of the second term.*

Study the application of I and II in the following

EXAMPLES

1. $(a+1)^2 = a^2 + 2a + 1$. 3. $(4+x)^2 = 16 + 8x + x^2$.
2. $(a-3)^2 = a^2 - 6a + 9$. 4. $(3-y)^2 = 9 - 6y + y^2$.
5. $(3x+a)^2 = 9x^2 + 6ax + a^2$.

ORAL EXERCISES

Expand the following by I or II of page 134:

1. $(x+1)^2$.
2. $(a+2)^2$.
3. $(x-5)^2$.
4. $(y+7)^2$.
5. $(2-x)^2$.
6. $(10-a)^2$.
7. $(4+x)^2$.
8. $(a-6)^2$.
9. $(10-y)^2$.
10. $(a+t)^2$.
11. $(x-t)^2$.
12. $(m-y)^2$.
13. $(x-2\,t)^2$.
14. $(3\,t-a)^2$.
15. $(5\,x+m)^2$.
16. $(3\,a-y)^2$.
17. $(7\,x-t)^2$.
18. $(11\,m+a)^2$.
19. $(x+.2)^2$.
20. $(t-.3)^2$.
21. $(3\,t+.4)^2$.
22. $(t+.05)^2$.
23. $(2\,t-.09)^2$.
24. $(.5-4\,r)^2$.

State two equal binomials whose product is:

25. $x^2+2\,x+1$.
26. $x^2+6\,x+9$.
27. $a^2-4\,a+4$.
28. $t^2-10\,t+25$.
29. $t^2-14\,t+49$.
30. $36-12\,x+x^2$.

Expand and find the value of:

31. $(8+3)^2$.

Solution. $(8+3)^2=8^2+2\cdot 8\cdot 3+3^2=64+48+9=121$.

32. $(7+1)^2$.
33. $(9+3)^2$.
34. $(11+2)^2$.
35. $(11-5)^2=(11)^2-2\cdot 11\cdot 5+5^2=121-110+25=36$.
36. $(10-3)^2$.
37. $(11-4)^2$.
38. $(12-3)^2$.
39. $(21)^2$.

HINT. $(21)^2=(20+1)^2$, etc.

40. $(31)^2$.
41. $(22)^2$.
42. $(41)^2$.
43. $(52)^2$.
44. $(63)^2$.
45. $(71)^2$.
46. $(85)^2$.
47. $(92)^2$.
48. $(101)^2$.
49. $(19)^2$.

HINT. $(19)^2=(20-1)^2$, etc.

50. $(38)^2$.
51. $(29)^2$.
52. $(78)^2$.
53. $(199)^2$.

EXERCISES

Expand the following:

1. $(2x+3)^2$.
2. $(3x-5)^2$.
3. $(2t-4m)^2$.
4. $(4y-3t)^2$.
5. $(10x-3y)^2$.
6. $(7a+6b)^2$.
7. $(8a+11b)^2$.
8. $(3x^2+8t)^2$.
9. $(3x^2-5x)^2$.
10. $(4x^3-7x)^2$.
11. $(3ax+4at)^2$.
12. $(-2a+3bx)^2$.
13. $(5xy-3xz)^2$.
14. $(10x^2y-7xy^2)^2$.
15. $(.5a+4t)^2$.
16. $(.4ax+6a)^2$.
17. $(.4t-.8x)^2$.
18. $(10x-.3t)^2$.

What changes must be made in the following so that the numerator will be the square of the denominator?

19. $\dfrac{a^2+ab+b^2}{a+b}$.
20. $\dfrac{a^2-at+t^2}{a-t}$.
21. $\dfrac{a^2+10a+100}{a+10}$.
22. $\dfrac{x^2+6x-9}{x+3}$.
23. $\dfrac{x^2-18x+81}{x+9}$.
24. $\dfrac{t^2-4t+16}{t-4}$.
25. $\dfrac{a^4-20a^2+25}{a^2-5}$.
26. $\dfrac{a^2-7a+49}{a+7}$.
27. $\dfrac{9x^2-6x+4}{3x-2}$.
28. $\dfrac{x^2-20x+25}{2x-5}$.
29. $\dfrac{x^2-24x+9}{4x-3}$.
30. $\dfrac{4a^2+20at+25t^2}{5t-2a}$.
31. $\dfrac{9x^2-12ax+a^2}{2a-3x}$.
32. $\dfrac{16x^2-24ax+a^2}{3a-4x}$.
33. $\dfrac{x^2+.8x+.16}{x-.4}$.
34. $\dfrac{4t^2-.4t+.01}{2t+.1}$.

66. The product of the sum and the difference of two terms. The multiplication

$$\begin{array}{lll} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

gives the formula $(a+b)(a-b)=a^2-b^2$.

This may be expressed in words as follows:

III. *The product of the sum and the difference of two terms equals the difference of their squares taken in the same order as the difference of the terms.*

Study the application of III in the following

EXAMPLES

1. $(a+2)(a-2)=a^2-4$.
2. $(7-x)(7+x)=49-x^2$.
3. $(m+t)(m-t)=m^2-t^2$.
4. $(4a-t)(4a+t)=16a^2-t^2$.
5. $(2x^2+7)(2x^2-7)=4x^4-49$.

ORAL EXERCISES

Perform the indicated operation:

1. $(x+3)(x-3)$.
2. $(a+5)(a-5)$.
3. $(t-6)(t+6)$.
4. $(1+a)(1-a)$.
5. $(t+x)(t-x)$.
6. $(1+3x)(1-3x)$.
7. $(x+5t)(x-5t)$.
8. $(4a-t)(4a+t)$.
9. $(7x+a)(7x-a)$.
10. $(11a-t)(11a+t)$.
11. $(a^2+12)(a^2-12)$.
12. $(4x-3y)(4x+3y)$.
13. $(2a+7t)(2a-7t)$.
14. $(4am-3)(4am+3)$.
15. $(ax+7)(ax-7)$.
16. $(a^2+5)(a^2-5)$.
17. $(a^4+12)(a^4-12)$.
18. $(xy-y^2)(xy+y^2)$.

19. $(8at - t^2)(8at + t^2)$.

20. $(.5x - t)(.5x + t)$.

21. $(9a - .7t)(9a + .7t)$.

22. $(x^2 - 4) \div (x - 2)$.

23. $(9x^2 - 1) \div (3x + 1)$.

24. $(49a^4 - 36) \div (6 + 7a^2)$.

State two binomials whose product is:

25. $a^2 - 4$.

26. $t^2 - 25$.

27. $36 - x^2$.

28. $y^2 - 64$.

29. $49 - y^4$.

30. $81 - x^2$.

31. $.01 - t^2$.

32. $a^2 - .09$.

33. $m^2 - .0016$.

34. $.16a^2 - 25$.

35. $x^2 - 9t^2$.

36. $16a^2 - 25y^2$.

37. $100a^2 - 1$.

38. $64a^2x^2 - 25$.

39. $1 - 36m^2$.

40. $25 - 49a^2$.

41. $100m^2 - 16$.

42. $36x^2y^4 - z^2$.

Find the value of:

43. $(11 + 3)(11 - 3)$.

44. $(20 - 1)(20 + 1)$.

45. $(30 + 2)(30 - 2)$.

46. $(50 - 1)(50 + 1)$.

47. $21 \cdot 19$.

HINT. $21 \cdot 19 = (20 + 1)(20 - 1)$, etc.

48. $22 \cdot 18$.

49. $29 \cdot 31$.

EXERCISES

Perform the indicated operations:

1. $[x + y + t][x + y - t]$.

Solution. This is similar to $(a + b)(a - b) = a^2 - b^2$ if we regard $x + y$ as replacing a, and t as replacing b. Inclosing $x + y$ in parentheses to indicate this we have

$$\begin{aligned} [x + y + t][x + y - t] &= [(x + y) + t][(x + y) - t] \\ &= (x + y)^2 - t^2 \\ &= x^2 + 2xy + y^2 - t^2. \end{aligned}$$

2. $[(x + y) + 2][(x + y) - 2]$.

3. $[x + 3 - a][x + 3 + a]$.

4. $[x + 4 + c][x + 4 - c]$.

5. $[(2a - b) + c][(2a - b) - c]$.

6. $[3x - t - 5c][3x - t + 5c]$.

7. $[2t + a - 4x][2t + a + 4x]$.

8. $[5a - b + 2c][5a - b - 2c]$.

9. $[a + x + y][a - x - y]$.

Solution.
$$\begin{aligned} [a+x+y][a-x-y] &= [a + (x + y)][a - (x + y)] \\ &= a^2 - (x + y)^2 \\ &= a^2 - (x^2 + 2xy + y^2) \\ &= a^2 - x^2 - 2xy - y^2. \end{aligned}$$

10. $[t + x + a][t - x - a]$.

11. $[t + a + 3][t - a - 3]$.

12. $[t + a - x][t - a + x]$.

HINT. This may be written $[t + (a - x)][t - (a - x)]$, etc.

13. $[x - a + 2][x + a - 2]$.

14. $[3 + x + y][3 - x - y]$.

15. $[4a + 2y - x][4a - 2y + x]$.

16. $[10 - t + 5s][10 + t - 5s]$.

67. The product of two binomials having a common term. The multiplication

$$\begin{array}{l} x + a \\ \underline{x + b} \\ x^2 + ax \\ \underline{\quad\;\; + bx + ab} \\ x^2 + (a + b)x + ab \end{array}$$

gives the formula

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

This may be expressed in words as follows:

IV. *The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.*

Study the application of IV in the following

EXAMPLES

1. $(x+2)(x+3)=x^2+(2+3)x+6=x^2+5x+6.$
2. $(x-5)(x-3)=x^2+(-5-3)x+15=x^2-8x+15.$
3. $(x+4)(x-7)=x^2+(4-7)x-28=x^2-3x-28.$

ORAL EXERCISES

Perform the indicated operations:

1. $(a+1)(a+3).$
2. $(a+3)(a+4).$
3. $(a+2)(a+4).$
4. $(x+5)(x+4).$
5. $(x+7)(x+3).$
6. $(x+5)(x+6).$
7. $(t+4)(t+7).$
8. $(a-1)(a-4).$
9. $(a-2)(a-5).$
10. $(a-3)(a-4).$
11. $(a-5)(a-3).$
12. $(a-6)(a-5).$
13. $(a-5)(a-7).$
14. $(x-7)(x-8).$
15. $(x-5)(x+2).$
16. $(x-6)(x+3).$
17. $(x+4)(x-5).$
18. $(x-8)(x+4).$
19. $(t+a)(t+b).$
20. $(t+m)(t-a).$
21. $(x-b)(x+s).$
22. $(x^2+3x+2)\div(x+2).$
23. $(x^2+5x+6)\div(x+2).$
24. $(x^2+7x+10)\div(x+5).$
25. $(a^2-6a+8)\div(a-4).$
26. $(a^2-a-6)\div(a-3).$
27. $(a^2-a-12)\div(a-4).$
28. $(a^2+a-20)\div(a+5).$
29. $(a^2+a-42)\div(a-6).$
30. $(t^2-at-6a^2)\div(t-3a).$
31. $(a^2+4at-21t^2)\div(a-3t).$
32. $(x^2+7xy-18y^2)\div(x+9y).$
33. $(t^2-at-90a^2)\div(t-10a).$

Supply the missing term or sign and then carry out the following as exact divisions:

34. $(t^2-t-30)\div(t+?)$.	**39.** $(x^2-3x-40)\div(x\,?\,5)$.
35. $(t^2+t-56)\div(t+?)$.	**40.** $(x^2+3x-28)\div(x\,?\,4)$.
36. $(a^2-4a-32)\div(a-?)$.	**41.** $(x^2-5x-24)\div(x+?)$.
37. $(a^2+2a-48)\div(a-?)$.	**42.** $(x^2+3x-10)\div(x\,?\,5)$.
38. $(a^2-a-72)\div(a\,?\,9)$.	**43.** $(x^2-x-42)\div(x-?)$.

68. The product of two binomials of the form $(ax+b)(cx+d)$. The multiplications

$$\begin{array}{lll} 2x & + & 5 \\ 3x & - & 7 \\ \hline 6x^2 & + & 15x \\ & - & 14x-35 \\ \hline 6x^2 & + & x-35 \end{array} \quad \text{and} \quad \begin{array}{lll} ax & + & b \\ cx & + & d \\ \hline acx^2 & + & bcx \\ & + & adx+bd \\ \hline acx^2 & + & (bc+ad)x+bd \end{array}$$

show that in the product of any two binomials of the form $(ax+b)(cx+d)$

(*a*) the first term is the product of the first terms of the binomials,

(*b*) the second term is the sum of the cross products, and

(*c*) the third term is the algebraic product of the second terms of the binomials.

EXAMPLES

1. $(2x+3)(3x+4)=6x^2+9x+8x+12=6x^2+17x+12.$
2. $(3t+5)(4t-1)=12t^2+20t-3t-5=12t^2+17t-5.$
3. $(5a+2)(2a-7)=10a^2-31a-14.$

EXERCISES

Write the products of the following:

1. $(2a+3)(2a+1)$.
2. $(2a+2)(2a+3)$.
3. $(3t-1)(3t+4)$.
4. $(2t+3)(2t-5)$.
5. $(2t-3)(2t+5)$.
6. $(2x-3)(2x+4)$.
7. $(2x-1)(2x+5)$.
8. $(2a-3)(a-1)$.
9. $(2a-2)(a-3)$.
10. $(3t+4)(t-1)$.
11. $(3a-5)(a+2)$.
12. $(4x+3)(x-5)$.
13. $(4t-5)(t-3)$.
14. $(4t-1)(t+5)$.
15. $(3a-2)(a-4)$.
16. $(2a-1)(3a+2)$.
17. $(2x+3)(3x-1)$.
18. $(3x-2)(4x+3)$.
19. $(5t-6x)(4t+5x)$.
20. $(4a-3t)(5a-7t)$.
21. $(5x-2a)(7x+3a)$.
22. $(4x-9t)(2x-5t)$.

ORAL REVIEW EXERCISES

Perform the following indicated operations:

1. $(h+2k)^2$.
2. $(3h-k)^2$.
3. $(n-4h)^2$.
4. $(2m+3k)^2$.
5. $(m-4r)^2$.
6. $(m^2+3r)^2$.
7. $(n-3r)(n+3r)$.
8. $(h-4n)(h+4n)$.
9. $(3n^2h+2h)^2$.
10. $(5m^2-k^2)^2$.
11. $(3n-5h)(3n+5h)$.
12. $(5n^3-4n^2h)^2$.
13. $(7n^2-h)(7n^2+h)$.
14. $(-3n^2h-2h^2)^2$.
15. $(29)^2$.
16. $(71)^2$.
17. $(89)^2$.
18. $(n+7)(n+3)$.
19. $(n-8)(n-3)$.
20. $(n-6)(n+9)$.
21. $(7n^2r-3)(7n^2r+3)$.
22. $(t^2+16)(t^2-16)$.

23. $(4t^2 - 9)(4t^2 + 9)$.

24. $(n - t)(n^2 + t^2)(n + t)$.

HINT. Multiply the first binomial by the third and this result by the second binomial.

25. $(n - 3)(n^2 + 9)(n + 3)$.

26. $(t + 1)(t^2 + 1)(t^4 + 1)(t - 1)$.

27. $(n - 2)(n^2 + 4)(n^4 + 16)(n + 2)$.

28. $(m + 2)(m - 2)(m^2 - 4)$.

29. $(a - bc)(a + bc)(a^2 - b^2c^2)$.

30. $(x + 2)(x^2 + 4)(x - 2)(x^4 + 16)$.

31. $(x^2 + y^2)(x - y)(x + y)(x^4 - y^4)$.

32. $[n + r + t][n + r - t]$.

33. $[n + r - 5][n + r + 5]$.

34. $(2n + 3r)(n + r)$.

35. $(2n - 3r)(2n + r)$.

36. $(3r - 5n)(2r - n)$.

37. $(3rn - 2t)(4rn + t)$.

38. $(5r^2n + h)(2r^2n - 3h)$.

39. $(4k - 6nt)(7k + 2nt)$.

40. $(n^2 - 3t)(4n^2 + 5t)$.

41. $(4t^2 + h^2)(4t^2 - h^2)$.

42. $(4t^2 - 2n^3)(4t^2 + 3n^3)$.

43. $(n^2 + 2nt + t^2) \div (t + n)$.

44. $(n^2 - 20nt + 100t^2) \div (10t - n)$.

45. $(n^8 - 81) \div (n^4 + 9)$.

46. $(t^2 + 3t + 2) \div (t + 2)$.

47. $(n^2 - 5n + 6) \div (n - 3)$.

Indicate the simplest way to obtain the products of the following:

48. $(n + t)(n + t)(n - t)(n - t)$.

49. $(n^2 + r^2)(n + r)(n^4 + r^4)(n - r)$.

50. $(4t^2 + n^2)(2t + n)(16t^4 + n^4)(2t - n)$.

CHAPTER XIV

FACTORING

69. Definitions. *Factoring* is the process of finding two or more expressions whose product is equal to a given expression.

Many simple exercises in factoring were solved in the preceding chapter in connection with the rules of multiplication there given. In fact, the process of factoring is the reverse of multiplication.

The subject of factoring is extensive. In this chapter we shall consider only the more common forms of factorable expressions, and usually seek only such factors as have integers as coefficients.

If a polynomial cannot be expressed as the product of expressions other than itself and 1, it is said to be *prime*.

Thus $2x + 1$, $3a - 5$, and $x^2 + 1$ are prime.

70. Square root of monomials. In factoring it is often necessary to find the square root, the cube root, and other roots of monomials.

The *square root* of a monomial is one of the two equal factors whose product is the monomial.

Since $+2 \cdot +2 = 4$ and $-2 \cdot -2 = 4$, the square root of 4 is ± 2, which means both plus 2 and minus 2.

Similarly, the square root of 9 is ± 3 and the square root of a^2 is $\pm a$.

That is, *Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.*

It is customary to speak of the positive square root of a number as the *principal square root,* and if no sign precedes the radical, the principal root is understood.

Thus, $\sqrt{4} = 2$, not -2; $-\sqrt{4} = -2$, not $+2$.

When both the positive and the negative square roots are considered, the double sign must precede the radical.

Since $x^3 \cdot x^3 = (-x^3)(-x^3) = x^6$, then $\pm\sqrt{x^6} = \pm x^3$.

That is, *The exponent of any letter in the square root of a monomial is one half the exponent of that letter in the monomial.*

Hence for extracting the square root of a monomial where both positive and negative factors are desired we have the

Rule. ***Write the square root of the numeric coefficient preceded by the double sign ± and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.***

A rule similar to the preceding one holds for the fourth root, sixth root, and other even roots.

Thus, $\pm\sqrt[4]{81\,c^8} = \pm 3\,c^2$, and $\pm\sqrt[6]{a^{18}} = \pm a^3$.

In this chapter, and also in Chapter XVI on Fractions, where square roots arise, only the *principal* square root will be considered.

According to the definition of square root, the two factors of a term either of which is its square root *must be equal.* Consequently they must have the same sign.

ORAL EXERCISES

Perform the indicated operation in the following:

1. $\sqrt{2^2}$.
2. $\sqrt{5^2}$.
3. $\sqrt{3^4}$.
4. $\sqrt{2^2 \cdot 3^2}$.
5. $\sqrt{4^2 \cdot 7^2}$.
6. $\sqrt{4\,a^2}$.
7. $\sqrt{9\,x^4}$.
8. $\sqrt{16\,x^2}$.
9. $\sqrt{25\,x^6}$.
10. $\sqrt{9\,a^2x^2}$.
11. $\sqrt{4\,x^2a^4}$.
12. $\sqrt{36\,a^2x^6}$.
13. $\sqrt{64\,a^2x^4}$.
14. $\sqrt{49\,a^4x^6}$.
15. $\sqrt{81\,a^8x^2}$.
16. $\sqrt{169\,a^6x^2}$.
17. $\sqrt{144\,a^2x^8}$.
18. $6\sqrt{100\,a^{10}}$.
19. $5\sqrt{121\,a^{20}}$.
20. $2\sqrt{196\,a^{12}x^4}$.
21. $\sqrt{225\,a^4x^8}$.
22. $\sqrt{2^2 \cdot 3^2 \cdot 7^2}$.
23. $\sqrt{9 \cdot 25 \cdot 81}$.
24. $\sqrt{3^2 \cdot 16 \cdot 5^2}$.
25. $\sqrt{4\,x^2(a^2)^2}$.
26. $3\sqrt{x^4(y^3)^2}$.
27. $ax\sqrt{(x+a)^2}$.
28. $3\sqrt{4(x-a)^2}$.
29. $\sqrt{9\,x^2(x+a)^2}$.
30. $2\sqrt{(3\,x-2)^2a^4}$.

71. Polynomials with a common monomial factor. The type form is

$$ab + ac - ad.$$

Since $$ab + ac - ad = a(b + c - d),$$

we have, for factoring expressions having a common monomial factor, the following

Rule. *Determine by inspection the monomial factor which is the product of all numeric and literal factors common to every term of the polynomial.*

Divide the polynomial by this monomial factor.

Write the quotient in a parenthesis preceded by the monomial factor.

EXAMPLE

Factor $6\,a^3x - 15\,a^2$.

Solution. The common monomial factor of both terms is seen to be $3\,a^2$. Dividing the binomial by $3\,a^2$ the quotient is $2\,ax - 5$.

Therefore $$6\,a^3x - 15\,a^2 = 3\,a^2(2\,ax - 5).$$

ORAL EXERCISES

Factor the following:

1. $2x - 2$.
2. $2x + 4$.
3. $3x - 6$.
4. $5x - 10$.
5. $ax + a$.
6. $3ax - 6a$.
7. $7a - 21$.
8. $9 - 3a$.
9. $5x^2 + 10x$.
10. $cx - 2c$.
11. $ax - cx$.
12. $ac - c^2$.
13. $3ax - 6a$.
14. $4a^2 - 12a$.
15. $2\pi R - 2\pi r$.
16. $ax + acx$.
17. $3ax - 6ax^2$.
18. $5ax^2 - 10x^2$.
19. $12ax - 10bx$.
20. $2a^2x + ax^2$.
21. $2ac + abc$.
22. $3ax - 6acx$.
23. $4ac - 12a$.
24. $ax - bx^2$.
25. $a^2x + ax$.
26. $2ax - 4ab$.
27. $3ax + 6bx$.
28. $2ax^2 + a^2x$.
29. $acx - 3ac + c$.
30. $am + an + ax$.

EXERCISES

Write the prime factors of:

1. $2a + 6$.
2. $ax + x^2$.
3. $a^3 + a^2$.
4. $3ax - 15a^2$.
5. $2a^3 + 6a$.
6. $2\pi r - 2\pi a$.
7. $a^2b - ab^2$.
8. $2\pi Rl + 2\pi rl$.
9. $3ax - 6ay$.
10. $4a^2 + 16ab$.
11. $ab - ax - ac$.
12. $\pi R^2h + \pi r^2h + \pi Rrh$.
13. $2a - 4a^2 + 6a^3$.
14. $a^5 + a^2 - a^4$.
15. $3c^2 - 12c - 18c^4$.
16. $ax - a^2x^2 - a^3x$.
17. $ay - abc - aby$.
18. $a^2 - 2ax + a$.
19. $3c^2 - 15c + 6c^3$.
20. $a^4 - a^3 + a^2 + a$.
21. $8a^2 - 4a^6 + 12a^3 - 6a^5$.
22. Solve for x, $ax = a(b + c)$.
23. Solve for x, $ax = am - ac$.
24. Solve for y, $my = ma + bm + cm$.
25. Solve for y, $ay = ab - abc - a^2$.
26. Solve for a, $ax = bx - cx + x^2$.
27. Solve for z, $az = a^2 - abc - 2a$.

Factor, then compute the numeric value of:

28. $3 \cdot 5 \cdot 17^2 - 3 \cdot 5 \cdot 4^3$.

HINT. $3 \cdot 5 \cdot 17^2 - 3 \cdot 5 \cdot 4^3 = 3 \cdot 5(17^2 - 4^3)$, etc.

29. $(3.1416)196 - (3.1416)121$.

30. $7 \cdot 13(31)^2 + 5 \cdot 13(31)^2 - 3 \cdot 13(31)^2$.

31. $\frac{22}{7}(21)^2 + \frac{22}{7}(14)^2 - \frac{22}{7}(21 \cdot 14)$.

32. $3.1416 \cdot (13)^2 + 3.1416(65) + \frac{4(3.1416)(13)(11)}{2}$.

72. Polynomials having a common binomial factor. The type form is

$$ax + ay + bx + by.$$

Plainly, $ax + ay + bx + by = a(x + y) + b(x + y)$.

Dividing both terms of $a(x + y) + b(x + y)$ by $(x + y)$, the quotient is $a + b$.

Therefore $ax + ay + bx + by = (x + y)(a + b)$.

EXAMPLE

Factor $2ac + 5bx + bc + 10ax$.

Solution. $2ac + bc + 5bx + 10ax = c(2a + b) + 5x(b + 2a)$

$= (2a + b)(c + 5x)$.

The preceding example illustrates the

RULE. *Arrange the terms of the polynomial to be factored in groups of two or more terms each such that in each group a monomial factor may be written outside a parenthesis, which in each case will contain the same expression.*

Rewrite, placing these monomial factors outside parentheses.

Then divide by the expression in parenthesis and write the divisor as one factor and the quotient as the other.

Polynomials which may be factored by grouping terms according to the foregoing rule usually contain either four, six, or eight terms.

It is important to note that one can obtain two apparently different sets of factors for a given expression. Thus

$$(a - b)(x - 3y) = (b - a)(3y - x),$$

for each pair by actual multiplication gives

$$ax - bx - 3ay + 3by.$$

An inspection of the expression shows that the two binomials of the first pair are the negatives respectively of those in the second pair; hence either pair of factors is correct.

The relation that the process of factoring bears to the processes of multiplication and division of monomials and polynomials should be kept constantly in mind. In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and the insight which can be obtained only by experience will enable us to determine the factors. To gain this insight rapidly one must keep clearly in mind the typical products of Chapter XIII and must study carefully certain typical forms, two of which have already been presented on pages 146 and 148.

There is no simple operation the performance of which makes us sure that we have found the *prime* factors of a given expression. Here, again, only the ability which comes with practice and experience enables us to find prime factors with certainty.

A partial check, however, that may be applied to all the exercises in factoring consists in actually multiplying together the factors that have been found. The result should be the original expression.

ORAL EXERCISES

1. Divide $3(a-b)+c(a-b)$ by $a-b$. What is the quotient? What are the factors of the first expression?

Give the binomial factors for:

2. $3(a-b)+c(a-b)$.
3. $a(x+2)+b(x+2)$.
4. $x(a-2b)+y(a-2b)$.
5. $a(m-2n)-b(m-2n)$.

Factor:

6. $x(a-b)+y(a-b)$.
7. $2n(a-7)+(a-7)$.
8. $(c-a)a-(c-a)c$.
9. $(s+t)x+(t+s)y$.
10. $a(n-7)-b(-7+n)$.
11. $x(r-t)+2y(-t+r)$.
12. $b(a-x)-(a-x)$.
13. $2m(a-2t)+(a-2t)$.
14. $3m(a-2c)-n(a-2c)$.
15. $(x-y)m+n(x-y)$.
16. $r(t-x)-(t-x)m$.
17. $2a(x-m)+(x-m)5b$.
18. $4a(2x-7)-3(2x-7)$.
19. $2a(a-3c)+(a-3c)x$.
20. $7x(a-b)-3(a-b)$.
21. $a(x-y)+b(x-y)-c(x-y)$.
22. $x(a-3)+c(a-3)+(a-3)$.

EXERCISES

Factor the following:

1. $2m + bm + 2z + bz$.

HINT. Collecting the coefficients of m and z we have

$(2 + b)m + (2 + b)z$, etc.

2. $ax + 2x + ay + 2y$.
3. $rx + ry + sx + sy$.
4. $ax + ay + 2x + 2y$.
5. $ay + az + cy + cz$.
6. $ah + bh + ak + bk$.
7. $x^3 + x^2 + x + 1$.
8. $a^3 + 1 + a + a^2$.
9. $4x(a - c) + 3(c - a)$.

HINT. This can be written

$4x(a - c) - 3(a - c)$, etc.

10. $t(c - t) - x(t - c)$.
11. $rx + x + ar + a$.
12. $ac + ax + x + c$.
13. $28ac + 9rx + 21ar + 12cx$.
14. $2ax + 7a^2 + 16bx + 56ab$.
15. $36by + 45bd + 4y + 5d$.
16. $5x^3 + 10x + 5x^2 + 10$.
17. $3b + 5by^3 + 6hy + 10hy^4$.
18. $a(r - t) - c(t - r)$.
19. $2x(c - t) + 3(t - c)$.
20. $3ax - 6bx - ay + 2by$.

Solution. $3ax - 6bx - ay + 2by$

$3x(a - 2b) - y(a - 2b) = (a - 2b)(3x - y)$.

21. $ax + 3a - bx - 3b$.
22. $6ax - 2ac + 3x - c$.
23. $ac - ax + bx - bc$.
24. $2an + 9bm - 3am - 6bn$.
25. $m^2 - at + amt - m$.
26. $6rx - 4rc - 6c + 9x$.
27. $a^3 - 2ax - a^2x + 2x^2$.
28. $a^2c - acx + acx^2 - a^2cx$.
29. $r^3 + at^2 + r^2at + rt$.
30. $7he - 77hz - de + 11dz$.
31. $12ax^5 - 6ax^3 - 4x^5 + 2x^3$.
32. $4dr - 4rs + dst - d^2t$.

33. $16\,acxy - 40\,bx^2y - 6\,abc^2 + 15\,b^2cx$.

34. $36\,axy + 45\,acy - 4\,xy - 5\,cy$.

35. Solve for x, $x(a+1) = b(a+1) + c(a+1)$.

36. Solve for x, $ax + bx = ac + bc + 2\,a + 2\,b$.

HINT. In the preceding equation, and in similar ones, the value of the unknown should be obtained by the use of factoring. In the final step of the solution mental division should be employed, not ordinary long division.

37. Solve for z, $abz + z = abr + r + 2\,ab + 2$.

38. Solve for y, $ay - 3\,y = am - 3\,m - 6 + 2\,a$.

39. Solve for x, $2\,ax + cx = 2\,ab + cm + bc + 2\,am$.

40. Solve for x, $ax + bc = a^2 + ab - ac - bx$.

41. Solve for x, $ax + 2\,c = ac + 3\,a - 6 + 2\,x$.

73. Trinomials which are perfect squares. Here the type form is

$$a^2 \pm 2\,ab + b^2.$$

This, as on page 134, gives us the two expressions:

$$a^2 + 2\,ab + b^2 = (a+b)^2, \quad a^2 - 2\,ab + b^2 = (a-b)^2.$$

If an algebraic expression is the product of two equal factors, it is said to be a *perfect square.*

A *trinomial,* arranged according to the descending powers of one letter, *is a perfect square if the first and third terms are positive and if the absolute value of the middle term is twice the product of the absolute values of the square roots of the other two terms.*

Thus, in the type form above, the middle term $2\,ab = 2 \cdot \sqrt{a^2} \cdot \sqrt{b^2}$.

Similarly, the trinomial $25\,a^4 + 60\,a^2b + 36\,b^2$ is a perfect square, since the middle term $60\,a^2b = 2 \cdot \sqrt{25\,a^4} \cdot \sqrt{36\,b^2} = 2 \cdot 5\,a^2 \cdot 6\,b$.

ORAL EXERCISES

Form perfect trinomial squares by supplying the missing term in:

1. $a^2 + (?) + 1$.
2. $a^2 + (?) + m^2$.
3. $a^2 + (?) + 4$.
4. $x^2 + (?) + 25$.
5. $x^2 + (?) + 9t^2$.
6. $1 + (?) + 16c^2$.
7. $x^2 + 2cx + (?)$.
8. $x^2 + 4ax + (?)$.
9. $x^2 + 10ax + (?)$.
10. $a^2 - 6ax + (?)$.
11. $n^2 - 18n + (?)$.
12. $a^2 - 14ax + (?)$.
13. $c^2 + 8c + (?)$.
14. $c^2 - 5ac + (?)$.
15. $x^4 + 12x^2c + (?)$.
16. $x^4 - 12x^2 + (?)$.
17. $4x^2 + 4x + (?)$.
18. $9x^2 - 6x + (?)$.
19. $4x^2 - 20cx + (?)$.
20. $16a^2 - 16ax + (?)$.
21. $9x^2 - 18ax + (?)$.
22. $9x^2 + 24cx + (?)$.
23. $16x^2 - 32mx + (?)$.
24. $(?) + 12x + 36$.
25. $(?) - 16x + 64$.
26. $(?) - 24ac + 9c^2$.
27. $16x^2 - (?) + 4y^2$.
28. $25a^2 + 10a + (?)$.

For obtaining *one* of the two equal factors of a perfect trinomial square we have the

Rule. *Arrange the terms of the trinomial according to the descending powers of some letter in it.*

Extract the square roots of the first and third terms and connect the results by the sign of the middle term.

Before applying the foregoing rule one should never fail to observe whether the expression to be factored is a perfect trinomial square or not.

ORAL EXERCISES

Factor the following:

1. $x^2 - 4x + 4$.
2. $x^2 + 6x + 9$.
3. $x^2 - 10x + 25$.
4. $a^2 - 12a + 36$.
5. $4a^2 - 4a + 1$.
6. $9a^2 + 6a + 1$.
7. $9c^2 - 12c + 4$.
8. $a^2 - 14a + 49$.
9. $9x^4 - 18x^2 + 9$.

10. $c^2 + 4 - 4c$.
11. $x^2 + 9 - 6x$.
12. $36t^2 - 12t + 1$.
13. $64 - 16m^2 + m^4$.
14. $81 - 18t^3 + t^6$.
15. $100 - 20t + t^2$.
16. $1 + 6t^2 + 9t^4$.
17. $a^6 + 4a^3 + 4$.
18. $t^4 - 10t^2 + 25$.
19. $c^8 + 2c^4 + 1$.
20. $9t^2 - 42t + 49$.
21. $25x^2 - 20x + 4$.
22. $9t^2 - 30t + 25$.
23. $121t^2 - 44t + 4$.
24. $t^{10} - .4t^5 + .04$.
25. $x^2 + x + \frac{1}{4}$.
26. $4x^2 + x + \frac{1}{16}$.
27. $9a^2 - 2a + \frac{1}{9}$.

EXERCISES

Factor the following:

1. $x^2 + 2xy + y^2$.
2. $a^2 - 2at + t^2$.
3. $a^2 - 10at + 25t^2$.
4. $16t^2 - 8at + a^2$.
5. $4x^2 - 12xy + 9y^2$.
6. $25x^2 - 20xy + 4y^2$.
7. $9a^2 - 30at + 25t^2$.
8. $4a^2 - 28at + 49t^2$.
9. $9t^2 - 12at + 4a^2$.
10. $x^2 + \frac{2}{3}x + \frac{1}{9}$.
11. $a^2 - 22ax + 121x^2$.
12. $144t^2 - 120at + 25a^2$.
13. $169x^2 + 78ax + 9a^2$.
14. $36m^2 + 25n^2 - 60mn$.
15. $x^4 + 4y^4 + 4x^2y^2$.
16. $4a^2t^8 - 4at^4 + 1$.
17. $12a^4t^3 + 4a^8 + 9t^6$.
18. $121x^2 - 110cx + 25c^2$.
19. $9x^4 - 66x^2 + 121$.
20. $\frac{25}{16}x^2 + \frac{5}{2}x + 1$.
21. $y^2 + y + \frac{1}{4}$.
22. $36a^2 + 12a + 1$.
23. $121x^2z^2 - 220txz + 100t^2$.
24. $169a^2 - 156az + 36z^2$.

Solve for x:

25. $(a + b)x = a^2 + 2ab + b^2$.
26. $ax - cx = a^2 - 2ac + c^2$.
27. $ax - 2tx = a^2 - 4at + 4t^2$.

Solve for z:

28. $nz - 4z = n^2 - 8n + 16$.

29. $az - c^2 = a^2 + 2ac - cz$.

30. $az + 2an = a^2 + nz + n^2$.

It is only in the beginning of factoring that polynomials are classified for the student. In the practical work of handling fractions and solving equations he must determine for himself the type of the polynomial to be factored. It is therefore very important that he fix in mind the various types and the manner of factoring each. Moreover, he should remember that the polynomials which arise in practice often have three or more factors. Miscellaneous review exercises afford excellent practice in recognizing types and in determining *all* the prime factors.

At this point the suggestions given on page 167 will prove helpful, though only the first two of the types there given have as yet been considered.

REVIEW EXERCISES

Separate into prime factors:

1. $3x^2 + 18x + 27$.
2. $a^3 + 2a^2 + a$.
3. $2t^4 - 12t^2 + 18$.
4. $t^5 - 4t^4 + 4t^3$.
5. $a^2x^2 - 2a^2x + a^2$.
6. $12t^2 + 36t + 27$.
7. $2c^3 - 20c^2 + 50c$.
8. $4x^2 - 16c^2x + 16c^4$.
9. $8a^3 - 40a^2t + 50at^2$.
10. $5at^4 - 70at^2 + 245a$.
11. $32at^8 - 48at^6 + 18at^4$.
12. $7t^4 + 28t^2 - 28t^3$.
13. $14ant - 7n - 21bnz$.
14. $2at + 2ax + 2bt + 2bx$.
15. $3ct - 3cx + 3at - 3ax$.
16. $at - a + t - 1$.

17. $x^4 + x^3 + x^2 + x$.

18. $2a^5 - 2a^4 + 2a^3 - 2a^2$.

19. $30at - 15a + 17c - 34ct$.

20. $6atx + 6btx - 10aty - 10bty$.

21. $56x^2 + 63xc - 40xt - 45ct$.

Solve for x:

22. $(a + b)x = c(a + b) - d(a + b)$.

23. $ax - bx = a^2 - 2ab + b^2$.

24. $ax - cx = ab + an - cn - bc$.

25. $ax + 3x = ab - 6t + 3b - 2at$.

26. $tx - 4c^2 = t^2 - 4ct + 2cx$.

27. $at + t + mx = am + m + tx$.

28. $2bct = bt^2 + bc^2 - bcx + btx$.

Factor:

29. $\pi R^2 - \pi r^2 + 4\pi(R + r)^2$.

30. $[\pi R^2 + \pi(R + r)^2 - \pi r^2]2h$.

31. $(\pi R^2 + \pi r^2)2h - (\pi R^2 + \pi r^2)h$.

32. $\pi h(R - r)^2 - \pi h(R + r)^2 + \pi hR^2$.

74. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

By page 137, $a^2 - b^2 = (a + b)(a - b)$.

Hence we have the

RULE. *Extract the square root of the absolute value of each term of the binomial.*

Add the two square roots for one factor, and subtract the second from the first for the other.

ORAL EXERCISES

Factor:

1. $a^2 - x^2$.
2. $x^2 - 1$.
3. $x^2 - 4$.
4. $4x^2 - 1$.
5. $a^2 - 9$.
6. $1 - m^2$.
7. $16 - t^2$.
8. $4t^2 - 9$.
9. $9t^2 - a^2$.
10. $\pi R^2 - \pi r^2$.
11. $9t^4 - 4x^2$.
12. $25t^2 - 36m^2$.
13. $49t^2 - 64r^2$.
14. $100x^4 - 81t^2$.
15. $100t^2 - 9x^2$.
16. $4\pi R^2 h - \pi r^2 h$.
17. $144t^2 - 49r^2$.
18. $64x^2 - 169t^2$.
19. $81r^2 - 121m^4$.
20. $.81t^2 - 1.96r^2$.
21. $169r^4 - .36t^2$.
22. $\frac{1}{25} - .04x^2$.
23. $.25 - \frac{1}{4}a^2$.
24. $m^2 - \frac{25}{64}$.
25. $2.25a^2 - 1.44b^2$.
26. $\frac{49}{36}x^2 - \frac{16}{25}y^2$.
27. $.36a^2b^2 - .25c^2$.
28. $x^2 - \frac{1}{9}$.
29. $\frac{4}{9}x^2 - 1.21$.
30. $\frac{4}{9}y^2 - .16z^2$.

Solve for x:

31. $x(t-2) = t^2 - 4$.
32. $x(3c+1) = 9c^2 - 1$.
33. $x(n-a) = n^2 - a^2$.
34. $(m-6)x = m^2 - 36$.
35. $(2t-3)x = 4t^2 - 9$.
36. $(4r - 5n)x = 16r^2 - 25n^2$.

EXAMPLES

1. Factor: $x^4 - 81$.

Solution. $x^4 - 81 = (x^2+9)(x^2-9)$, by application of the rule
$= (x^2+9)(x+3)(x-3)$, by application of the rule to $x^2 - 9$.

2. Factor: $16x^8 - 81a^{12}$.

Solution. $16x^8 - 81a^{12} = (4x^4 + 9a^6)(4x^4 - 9a^6)$
$= (4x^4 + 9a^6)(2x^2 + 3a^3)(2x^2 - 3a^3)$.

EXERCISES

Factor:

1. $a^4 - x^4$.

2. $a^4 - 1$.

3. $t^4 - 16$.

4. $x^4 - 81$.

5. $81 - t^8$.

6. $1 - 16\,m^4$.

7. $16 - a^8$.

8. $a^{12} - 81$.

9. $t^4 - 256$.

10. $16\,t^4 - 81$.

11. $x^4 - 625$.

12. $a^8 - b^4$.

13. $y^4 - a^8$.

14. $a^4 - 16\,b^4$.

15. $a^4t^8 - 81$.

16. $(a + x)^2 - t^2$.

HINT. $(a + x)^2 - t^2 = [(a + x) + t][(a + x) - t]$, etc.

17. $(a + t)^2 - 4$.

18. $(x - 3)^2 - m^2$.

19. $(2\,x + t)^2 - 9\,a^2$.

20. $4(x - 2\,a)^2 - 25$.

21. $9(t + 3\,x)^2 - 4\,m^2$.

22. $25\,a^2(x - t)^2 - 16\,m^2$.

23. $a^2 - (2\,x - t)^2$.

24. $m^2 - (3\,a - c)^2$.

25. $4\,x^2 - (2\,a + 3)^2$.

26. $16\,m^2 - (5\,a - x)^2$.

27. $25\,x^2 - (a + 7)^2$.

28. $a^2x^2 - (a - x)^2$.

29. $a^2t^2 - (t + a)^2$.

30. $9\,a^2 - 4(t - x)^2$.

31. $16\,a^2 - 9(m + t)^2$.

32. $25\,x^2 - (a - 2\,x)^2$.

33. $4\,r^2t^2 - (2\,r - t)^2$.

34. $36\,a^2t^2 - (at - 2\,x)^2$.

35. $16\,m^2 - 4\,a^2(x - 1)^2$.

36. $81\,t^2 - 16(2\,t - 3\,x)^2$.

37. $64\,x^2 - 9(x - 3\,t)^2$.

38. $(a + x)^2 - (b + y)^2$.

39. $(a - x)^2 - (b - t)^2$.

40. $(a - 2\,x)^2 - (3\,x + t)^2$.

41. $(2\,a - t)^2 - (2\,x - m)^2$.

42. $(3\,t - x)^2 - (m + 3\,a)^2$.

43. $(2\,a - 3\,b)^2 - (3\,c - 2\,x)^2$.

44. $(3\,x - 1)^2 - (1 - 2\,y)^2$.

Factor, then find the numeric value of:

45. $(12.7)^2 - (9.8)^2$.

Solution. $(12.7)^2 - (9.8)^2 = (12.7 + 9.8)(12.7 - 9.8)$

$= (22.5)(2.9) = 65.25$.

46. $(7.3)^2 - (4.3)^2$.

47. $(8.6)^2 - (5.6)^2$.

48. $(11.4)^2 - (6.4)^2$.

49. $(13.7)^2 - (3.7)^2$.

50. $(9.4)^2 - (6.3)^2$.

51. $(14.1)^2 - (7.8)^2$.

52. $7^2 - 2 \cdot 7 \cdot 5 + 5^2$.

53. $49 - 2 \cdot 7 \cdot 9 + 81$.

54. If $\pi = 3.1416$, $r = 7$, and $R = 11$, find the value of $\pi R^2 - \pi r^2$.

55. If $\pi = \frac{22}{7}$, $R = 4$, and $r = 3$, find the value of $\frac{4}{3}\pi R^3 - \frac{4\pi r^3}{3}$.

Some polynomials of four or six terms may be arranged as the difference of two squares and factored as in the preceding exercises.

EXERCISES

Factor:

1. $a^2 - 2ab + b^2 - x^2$.

Solution.
$$\begin{aligned} a^2 - 2ab + b^2 - x^2 &= (a-b)^2 - x^2 \\ &= (a-b+x)(a-b-x). \end{aligned}$$

2. $a^2 + 2at + t^2 - x^2$.

3. $a^2 + 2ac + c^2 - 4m^2$.

4. $a^2 - 4ax + 4x^2 - 9t^2$.

5. $4x^2 - 4x + 1 - m^2$.

6. $25 - 10t - 4m^2 + t^2$.

7. $x^2 - 16n^2 - 4tx + 4t^2$.

8. $y^2 - t^2 + x^2 - 2xy$.

9. $6xy - t^2 + y^2 + 9x^2$.

10. $4x^2 - 25y^2 + 9t^2 - 12tx$.

11. $x^2 - t^4 + 16m^2 - 8mx$.

12. $1 - 4at - t^2 + 4a^2t^2$.

13. $12ab - 4m^2 + 9a^2 + 4b^2$.

14. $9m^2 - 25n^2 - 6mt + t^2$.

15. $12at - 4a^2 - 9t^2 + m^2$.

Solution.
$$\begin{aligned} 12at - 4a^2 - 9t^2 + m^2 &= m^2 - (4a^2 - 12at + 9t^2) \\ &= m^2 - (2a - 3t)^2 \\ &= [m + (2a - 3t)][m - (2a - 3t)] \\ &= (m + 2a - 3t)(m - 2a + 3t). \end{aligned}$$

16. $m^2 - a^2 - 2\,at - t^2$.

17. $n^2 - b^2 - 2\,bd - d^2$.

18. $t^2 - b^2 + 4\,bc - 4\,c^2$.

19. $2\,tx - t^2 - x^2 + a^4$.

20. $4\,x^2 - a^2 + 4\,at - 4\,t^2$.

21. $6\,m + 9\,x^2 - 9 - m^2$.

22. $4\,ab - 4\,a^2 + 4\,m^6 - b^2$.

23. $a^2 + 2\,ab + b^2 - x^2 - 2\,tx - t^2$.

24. $x^2 - 4\,xt + 4\,t^2 - 9\,a^2 + 6\,a - 1$.

25. $1 + 2\,bc + 2\,a - c^2 - b^2 + a^2$.

26. $a^2 - 1 + t^2 - m^2 + 2\,at + 2\,m$.

27. $4\,a^2 + 10\,x - 25 - 12\,am - x^2 + 9\,m^2$.

REVIEW EXERCISES

Factor:

1. $a^3 - a$.

2. $t^4 - 2\,t^2 + 1$.

3. $x^4 - 8\,x^2 + 16$.

4. $a - a^5$.

5. $t^8 - 2\,t^4 + 1$.

6. $a^5 - 18\,a^3 + 81\,a$.

7. $a^3 - a + a^2x - x$.

8. $2\,a^3 - 18\,at^2$.

9. $t^4 - 10\,t^2 + 9$.

10. $t^3 + 2\,t^5 + t^7$.

11. $tn^5 - t^5n$.

12. $36\,a - 12\,a^3 + a^5$.

13. $\frac{1}{2}\,atm + \frac{1}{2}\,at$.

14. $t^3 - t^2 - 4\,t + 4$.

15. $3\,at + 3\,t^2 - 27\,t - 27\,a$.

16. $2\,a^3x + 3\,a^2x - 8\,ax - 12\,x$.

17. $na^2 - nt^2 + 4\,nt - 4\,n$.

18. $a^2x - 9\,x + a - 3$.

19. $2\,x^4 - 40\,x^2 + 200 - 18$.

20. $5\,a^4 + 20\,a^3 + 5\,a^5 + 20\,a^2$.

21. $a^2b^2 - b^2x^2 - 4\,a^2 + 4\,x^2$.

22. $\dfrac{1}{4} + r + r^2 - s^2 - \dfrac{2\,st}{3} - \dfrac{t^2}{9}$.

Solve for x:

23. $ax = a^3b - ab$.

24. $ab = ax - 4\,a^2t$.

25. $ax + bx = a^2 - b^2$.

26. $ax - 4\,t^2 = a^2 - 4\,at + 2\,tx$.

27. $t^2x - t^3 = tx - t$.

28. $x(t - 3)(t^2 + 9) = t^4 - 81$.

75. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

For many trinomials of this type two binomial factors may be found of the form $(x + r)(x + s)$. The method of factoring to be used is the reverse of the method of multiplying two binomials given on page 139. From a study of the four examples there given it is evident that to factor a trinomial of the form $x^2 + bx + c$ we must, if possible, find two numbers whose product is c and whose sum is b. Let these numbers be called r and s.

Then the required factors are $x + r$ and $x + s$,

since $$(x + r)(x + s) = x^2 + (r + s)x + rs.$$

EXAMPLES

1. Factor $x^2 + 12x + 32$.

Solution. $$x^2 + 12x + 32 = (x + ?)(x + ?).$$

It is necessary to find two numbers whose product is $+32$ and whose sum is $+12$.

Now $$32 = 1 \cdot 32 = 2 \cdot 16 = 4 \cdot 8.$$

The first two pairs are rejected, for each fails to give the sum $+12$. The third pair of factors of 32, namely 4 and 8, gives the correct sum.

Therefore $$x^2 + 12x + 32 = (x + 4)(x + 8).$$

2. Factor $a^2 - 11a + 24$.

Solution. Since 24 is positive, its two factors must have the *same* sign; since -11 is negative, both factors must be negative. Now $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$. By inspection of these products, -3 and -8 are found to be the required numbers.

Therefore $$a^2 - 11a + 24 = (a - 3)(a - 8).$$

3. Factor $c^2 - c - 42$.

Solution. The product -42 is negative; hence the required factors have *unlike* signs. Since -1 is negative, the negative factor of -42 must have the greater absolute value. Now $42 = 1 \cdot 42 = 2 \cdot 21 = 3 \cdot 14 = 6 \cdot 7$. We see that $6 + (-7) = -1$.

Therefore $$c^2 - c - 42 = (c + 6)(c - 7).$$

EXERCISES

Factor:

1. $x^2 + 3x + 2$.
2. $x^2 + 4x + 3$.
3. $x^2 + 5x + 4$.
4. $x^2 + 6x + 5$.
5. $x^2 + 7x + 6$.
6. $x^2 - 8x + 7$.
7. $x^2 - 9x + 8$.
8. $a^2 + 4a + 4$.
9. $a^2 + 5a + 6$.
10. $a^2 + 6a + 8$.
11. $a^2 + 7a + 10$.
12. $a^2 - 8a + 12$.
13. $a^2 - 9a + 14$.
14. $n^2 + 6n + 9$.
15. $n^2 + 7n + 12$.
16. $h^2 + 8h + 15$.
17. $x^2 - 9x + 20$.
18. $t^2 - 45 + 4t$.
19. $n^2 - 13n + 42$.
20. $t^2 - t - 6$.
21. $t^2 - t - 12$.
22. $x^2 + x - 20$.
23. $x^2 - x - 30$.
24. $h^2 - 3h - 10$.
25. $t^2 + 3t - 10$.
26. $t^2 - 2t - 15$.
27. $t^2 + 2t - 15$.
28. $t^2 - 4t - 21$.
29. $t^2 + 4t - 21$.
30. $t^2 - 4t - 45$.
31. $x^2 - 1.1x + .3$.
32. $t^2 + .5t + .06$.
33. $t^2 - .7t + .12$.
34. $x^2 + .2t - .15$.
35. $n^2 - 1n + .09$.
36. $x^2 - 1.6x - .8$.
37. $-x^2 + x + 20$.

Solution. $$-x^2 + x + 20 = -(x^2 - x - 20)$$
$$= -1(x - 5)(x + 4) = (5 - x)(x + 4).$$

38. $-n^2 - n + 12$.
39. $9 - 8n - n^2$.
40. $-2x - x^2 + 63$.
41. $80 - 2n - n^2$.
42. $t^2 + t^4 - 110$.
43. $a^2b^2 - 3aby - 28y^2$.
44. $n^6 - 11n^3 + 18$.
45. $a^2b^2 + 10abx^3 - 24x^6$.

46. $9tx^4y^6 + t^2x^8 - 22y^{12}$.

47. $33n^2 - a^2x^2 - 8anx$.

48. $(a+x)^2 + 3(a+x) + 2$.

49. $(a-x)^2 - 5(a-x) - 14$.

50. $n^2 + (a+b)n + ab$.

51. $n^2 + (2a+5)n + 10a$.

52. $(a-2t)^2 - 8(a-2t) + 15$.

53. $16 - 17(2t-x) + (2t-x)^2$.

REVIEW EXERCISES

Factor:

1. $x^3 - 4x$.
2. $4ax^2 - 4a$.
3. $a^4 + 6a^3 + 9a^2$.
4. $t^3 - 5t^2 + 6t$.
5. $3n^2 - 9n - 30$.
6. $3at^3 - 3at^2 - 6at$.
7. $x^4 - .5x^2 + .04$.
8. $nt^4 - 7nt^2 + 12n$.
9. $at^4 - 7at^2 - 18a$.
10. $a^4 - 13a^2 + 36$.
11. $3n^4 - 21n^3 - 54n^2$.
12. $56t + t^2 - t^3$.
13. $4ax^2 - 16a^3$.
14. $162a - 2a^5$.
15. $4t^5 + 4t^4 + t^3$.
16. $5an^4 - 20n^5$.
17. $t^8 - 17t^4 + 16$.
18. $81a - 18a^3 + a^5$.
19. $n^4 - 143n^2 - 144$.
20. $4a^5 - 92a^3 - 200a$.
21. $5an^4 - 5an^3 + 5an^2 - 5an$.
22. $4at^3 - 8at^2 - 24at + 48a$.
23. $10abc - 30bc + 20ac - 60c$.
24. $6at^2 + 3t^2c + 18atn + 9ctn$.
25. $4a^3 - 4ay^2 - 8ayz - 4az^2$.
26. Solve for x, $(a-3)x = a^2 - 5a + 6$.
27. Solve for x, $4x + ax = a^2 - 16$.
28. Solve for y, $ny + 42 = n^2 - n - 6y$.
29. Solve for y, $ay + a = a^2 + 5y - 20$.
30. Solve for z, $az + 3ac = a^2 + 2cz + 2c^2$.

Factor:

31. $\pi h(R+r)^2 + \pi h(R^2 - r^2)$.

32. $2\pi h(R^2 - r^2) + 3\pi h(R - r)^2$.

33. $5\pi(R^2h - 4h) + 2\pi(R+2)^2h$.

Simplify and factor:

34. $(\pi R^2 - \pi r^2)h + (\pi R^2 + \pi r^2)3h$.

35. $5\pi(R^2 - r^2)h - 2h(\pi R^2 - \pi r^2)$.

36. $[\pi R^2 + \pi r^2 + \pi(R+r)^2]h - [\pi R^2 + \pi r^2 + \pi Rr]h$.

76. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

For many trinomials of this type two binomial factors of the form $(hx+k)(mx+n)$ may be found. The method of factoring such trinomials is illustrated in the following

EXAMPLES

1. Factor $3x^2 + 7x + 2$.

Solution. $3x^2 + 7x + 2$

$= (?\,x + ?)(?\,x + ?)$.

To obtain the correct factors we must supply such numbers for the interrogation points in (1) and in (2) as will give

$$\begin{array}{rrrr} ?\,x & + \; ? & & \quad (1) \\ ?\,x & + \; ? & & \quad (2) \\ \hline 3x^2 & + \; ?\,x & & \\ & + \; ?\,x & + \; 2 & \\ \hline 3x^2 & + \; 7x & + \; 2 & \end{array}$$

$3x^2$ for the product of the first two terms of the binomials, $+2$ for the product of the last two terms of the binomials, and $+7x$ for the sum of the cross products.

Now $3x^2 = 3x \cdot x$,

and $+2 = 1 \cdot 2$.

The factors 1 and 2 may be substituted for the interrogation points in (1) and (2) in either of the following ways:

$$\begin{array}{r} 3x+2 \\ x+1 \end{array} \text{ (Incorrect)} \qquad \begin{array}{r} 3x+1 \\ x+2 \end{array} \text{ (Correct)}$$

The first pair is rejected, for it fails to give a product having the required middle term, $+7x$. The second pair gives the correct product.

Therefore $3x^2+7x+2=(3x+1)(x+2)$.

2. Factor $3x^2-13x-10$.

Solution. $3x^2=3x\cdot x$.

$$-10=1\cdot -10=-1\cdot 10=2\cdot -5=-2\cdot 5.$$

Test the following pairs of binomials:

$$\begin{array}{cccccccc} 3x+1 & x+1 & 3x-1 & x-1 & x+2 & 3x-2 & x-2 & 3x+2 \\ x-10 & 3x-10 & x+10 & 3x+10 & 3x-5 & x+5 & 3x+5 & x-5 \end{array}$$

Only the last pair gives the desired product.

Therefore $3x^2-13x-10=(x-5)(3x+2)$.

After a little practice it will usually be found unnecessary to write down all the pairs of binomials that do not produce the required product.

If none of the pairs gives the required product, the given trinomial is prime.

EXERCISES

Factor:

1. $2x^2+5x+2$.
2. $3x^2+5x+2$.
3. $2t^2+5t+3$.
4. $2x^2+7x+3$.
5. $3x^2+7x+2$.
6. $3a^2+10a+3$.
7. $2n^2+7n+6$.
8. $3t^2+8t+4$.
9. $2a^2+9a+9$.
10. $3x^4+11x^2+6$.
11. $2n^2-3n-2$.
12. $2t^2-5t-3$.
13. $2a^2-a-3$.
14. $2x^2-x-6$.
15. $3a^2+2a-1$.

16. $3\,t^2 - 5\,t - 2.$
17. $3\,n^2 - 8\,n - 3.$
18. $3\,a^2 + a - 2.$
19. $3\,t^2 + 4\,t - 4.$
20. $3\,n^4 + 7\,n^2 - 6.$
21. $3\,x^2 - 11\,x + 6.$
22. $3\,t^2 - 14\,t + 8.$
23. $3\,a^2 - 17\,a + 10.$
24. $6\,n^2 - n - 2.$
25. $6\,r^2 - 13\,r + 6.$
26. $10\,m^2 - 19\,m + 6.$
27. $15\,a^6 - 4\,a^3 - 4.$
28. $10\,n^2 - 39\,n + 14.$
29. $14 + 41\,t^2 + 15\,t^4.$
30. $6 + 35\,x^2 - 31\,x.$
31. $35\,a^2 + a - 6.$
32. $15\,t^2 - 24\,at + 9\,a^2.$
33. $11\,a^2b^2 - 80\,abx + 21\,x^2.$
34. $18\,a^2x^2 - 41\,anx - 10\,n^2.$
35. $35\,at^2 + 62\,abt - 33\,ab^2.$
36. $10\,an^2 - 59\,arn - 39\,ar^2.$
37. $6\,x^2 - 25\,x - 51.$
38. $12\,a^2t^2 - 53\,at + 30.$
39. $35\,x^6 - 102\,x^3 + 55.$
40. $26\,a^2n^2 + 81\,an - 35.$
41. $36\,x^2 - 9\,x - 10.$

77. General directions for factoring. Since no general method of factoring can be stated in a few simple rules, the process must be learned by means of such type forms and typical solutions as are given in the preceding pages. When these have been once thoroughly mastered, readiness in factoring expressions which are represented by them becomes a matter of experience. Generally a student finds it comparatively easy to factor a list of exercises classified under a particular type form, yet a list of miscellaneous exercises he finds difficult. This usually indicates inability to determine the type of an expression from its appearance. Until the student, by careful study of the type forms, has acquired the ability to do this, he will make little progress. There are many types that are not included in this book, which the student who continues the study of algebra will meet later.

The following suggestions will prove helpful in solving the types here considered:

I. *First look for a common monomial factor, and if there is one (other than 1) separate the expression into its greatest monomial factor and the corresponding polynomial factor.*

II. *Then by the form of the polynomial factor determine with which of the following types it should be classed and use the methods of factoring applicable to that type.*

1. $ax + ay + bx + by$.
2. $a^2 \pm 2ab + b^2$.
3. $a^2 - b^2$.
4. $x^2 + bx + c$.
5. $ax^2 + bx + c$.

III. *Proceed again as in II with each polynomial factor obtained until the original expression has been separated into its prime factors.*

MISCELLANEOUS EXERCISES

Factor:

1. $2n^3 - 2n$.
2. $3a^3 - 75a$.
3. $ax^4 - 81a$.
4. $\pi ar^2 - 2\pi rh$.
5. $\pi R^2 - 3\pi aR$.
6. $ax^4 - 16a$.
7. $a^4 - x^{16}$.
8. $2x^8 - 2$.
9. $3x^2 - 27y^2$.
10. $n^2 - n^6$.
11. $t^4 - 64t^8$.
12. $1 - 2n^2 + n^4$.
13. $n^8 - 2n^4 + 1$.
14. $16x - 8x^3 + x^5$.
15. $10a^2 - 10a - 60$.
16. $x^2 - .1x - .02$.
17. $9 + a^4 - 10a^2$.
18. $a^4 + 36 - 13a^2$.
19. $t^2 + 2n^2t^2 + n^4t^2$.
20. $\pi R^2 + \pi r^2 + 2\pi Rr$.
21. $.02a^2 - .3a - 2$.
22. $x^3 - x^2 - 90x$.
23. $4x^2 - 2xy - 30y^2$.
24. $4x^2 - 20tx + 25t^2$.

25. $14 n^2 + 16 n^4 + 3$.
26. $3 x^2 + 10 x - 8$.
27. $2 ab - a^2 - b^2$.
28. $72 x^2 - 96 xy + 14 y^2$.
29. $20 - x - x^2$.
30. $.5 x^2 - .3 x - .2$.
31. $3 a^2 + 5 a - 28$.
32. $4 n^4 - 9 n^2 - 9$.
33. $n^4 - 3 n^3 + 4 n^2 - 12 n$.
34. $2 a^3t + 3 a^2t - 8 at - 12 t$.
35. $.12 a^2 - .7 ab + b^2$.
36. $x^4 - x^2 - x + 1$.
37. $n^5 - n^4 - n^3 + n^2$.
38. $6 a^4n - 3 a^3bn - 3 a^2b^2n$.
39. $a^4 - a + 4 - 16 a^2$.
40. $1 + n^4 + n - n^2$.
41. $x^5 - x^3 + x - 1$.
42. $t^5 + t - t^4 - t^3$.
43. $n^4 - n^2 + 12 an - 36 a^2$.
44. $10 a^2 + 5 ab - 5 b^2$.
45. $x^2 - y^2 - x - y$.
46. $a - b + a^2 - b^2$.
47. $n^4 - 30 n^2 + 225 - 9 m^2$.
48. $289 - 100 x^2 - t^2 - 20 tx$.
49. $x^3 + 3 x^2 + 3 x + 9$.
50. $x^3 - 3 x^2 + 9 x - 27$.
51. $x^2 + 3.2 xy + .6 y^2$.
52. $n^3 + 3 n^2t + 3 nt^2 + 9 t^3$.
53. $18 n^3 - 6 n^2 + 3 n - 1$.
54. $4 ac - 2 c + 8 ad - 4 d$.
55. Solve for x, $a^2 + 3 x - ax - 9 = 0$.
56. Solve for t, $a^2 + c^2 + ct = at + 2 ac$.
57. Solve for z, $m^2 + 2 m + 3 z = mz + 15$.
58. Solve for x, $2 a^2 + 3 x = a + 2 ax + 3$.
59. Solve for t, $a^2 - 3 a - 2 t = at - 2 a + 6$.

CHAPTER XV

SOLUTION OF EQUATIONS BY FACTORING

78. Simple equations. A *simple* or *linear equation* in one unknown is one which may be put in such a form that

(*a*) the unknown does not appear in any denominator;
(*b*) only the first power of the unknown is involved.

Thus $5x - 2 = 13$, $3t - 7 = 5t - 21$, $ax + b = 0$, are simple equations. $(x + 4)(x - 5) = (x + 7)(x - 10)$ is also a simple equation, since on multiplying out it becomes $x^2 - x - 20 = x^2 - 3x - 70$, from which, after transposing, we get $2x + 50 = 0$.

In the preceding chapters only simple equations have been considered.

79. Quadratic equations. A *quadratic equation* in one unknown is one which may be put in such a form that

(*a*) the unknown does not appear in any denominator;
(*b*) the second but no higher power of the unknown is involved.

Thus $x^2 - 5x + 6 = 0$, $7x^2 + 4 = 6x + 16$, $ax^2 + bx + c = 0$ are quadratic equations.

A quadratic equation is often called an equation of the *second degree.*

The term in a quadratic equation which does not involve the unknown is called the *constant* term.

80. Solution of equations. The methods of factoring given in Chapter XIV enable us to solve many quadratic equations. In the solution of equations by factoring, use is made of the following

Principle. *If the product of two or more factors is zero, at least one of the factors must be equal to zero.*

Two or more, or even all, of the factors *may* be zero, but the vanishing of one is *sufficient* to make the product zero.

Consider the equation, in factored form,

$$(x - 3)(x - 8) = 0. \qquad (1)$$

If this equation is to be solved, all the numbers which satisfy it must be found. That is, we must find every value of x for which the product on the left of (1) is zero. If the product is to equal zero, the foregoing principle requires that one of the factors be zero. Hence any value of x which satisfies (1) must make either $x - 3 = 0$ or $x - 8 = 0$. Hence x must equal either 3 or 8. On substituting 3 for x in (1) we obtain $(3 - 3)(3 - 8) = 0$, or $0 \cdot (-5) = 0$. Hence 3 is a root of (1). On substituting 8 for x in (1) we obtain $(8 - 3)(8 - 8) = 0$, or $5 \cdot 0 = 0$. Hence 8 is also a root of (1). A moment's inspection makes it clear that 3 and 8 are the only roots of the equation.

ORAL EXERCISES

For what value of x is each of the following expressions equal to zero?

1. $x - 3$.
2. $x + 7$.
3. $x - 6$.
4. $x + 9$.
5. $2x - 10$.
6. $3x - 18$.
7. $2x - 5$.
8. $7x - 11$.
9. $ax - b$.

10. What is the value of 3×0? of 0×3? of -7×0? of $n \times 0$? of $m \times 0$?

11. Make a statement which will include all the results of Exercise 10.

Find the value of:

12. $(x-2)(x-3)$ when $x = 0, 2, 4$.
13. $(x-5)(x-6)$ when $x = 1, 3, 5$.
14. $(x+3)(x-1)$ when $x = 1, 3, -3$.
15. $(2x-1)(3x+1)$ when $x = -\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$.
16. $x(2x-8)$ when $x = 3, 0, 4$.

In the following determine which of the numbers on the right is a root of the corresponding equation:

17. $(x-2)(x-1) = 0$. $1, 2, 3$.
18. $(x-3)(x-4) = 0$. $-4, -3, 4, 3$.
19. $x(2x-5) = 0$. $1, \frac{5}{2}, 0$.
20. $(2t-3)(t+1) = 0$. $0, 1, -1, \frac{3}{2}$.
21. $(t-2)(t-3)(t-4) = 0$. $5, 2, 3, 4$.

What are the roots of the following equations?

22. $(x-2)(x+2) = 0$.
23. $x(x-3) = 0$.
24. $(x+3)(x-4) = 0$.
25. $(2t-1)(3t-2) = 0$.
26. $(3t+1)(5t-2) = 0$.
27. $(t-1)(t-3)(t-4) = 0$.
28. $(2t-1)(1-t)(t+2) = 0$.

29. Is there any one number which will make both factors of $(y-5)(y+7)$ equal to zero?

EXAMPLES

1. Solve $x^2 - 7x = 18$.

Solution. Transposing, $x^2 - 7x - 18 = 0$.

Factoring $(x - 9)(x + 2) = 0$.

The value of x which makes the first factor zero is a root of the quadratic. Setting $x - 9 = 0$, we obtain $x = 9$.

Similarly, that value of x which makes the second factor zero is also a root of the quadratic. Setting $x + 2 = 0$, we obtain $x = -2$.

Therefore $x = 9$ and $x = -2$ are the required roots.

Check. Substituting 9 for x in $x^2 - 7x = 18$ gives

$$81 - 63 = 18,$$

or

$$18 = 18.$$

Substituting -2 for x in $x^2 - 7x = 18$ gives

$$4 + 14 = 18,$$

or

$$18 = 18.$$

2. Solve $3x^2 = 5x$.

Solution. Transposing, $3x^2 - 5x = 0$.

Factoring, $x(3x - 5) = 0$.

Setting each factor equal to zero, we obtain $x = 0$ and $3x - 5 = 0$. Solving the second equation, $x = \frac{5}{3}$. Therefore the roots are $x = 0$ and $x = \frac{5}{3}$.

Check. $3 \times (0)^2 = 5 \times 0$, or $0 = 0$.

$$3(\tfrac{5}{3})^2 = 5(\tfrac{5}{3}), \text{ or } \tfrac{25}{3} = \tfrac{25}{3}.$$

For solving an equation in one unknown by factoring we have the

RULE. *Transpose the terms so that the right member is zero. Then factor the expression on the left, set each factor which contains the unknown equal to zero, and solve the resulting equations.*

It must be kept in mind that a root of an equation is a number which satisfies the equation.

One should never divide each member of an equation by an expression containing the unknown, for if this is done one or more roots may be lost.

Thus, if in Example 2 we had divided both sides of the equation by x, the resulting equation would have been $3x - 5 = 0$, which, to be sure, gives us one root of the given equation. But we have lost the root $x = 0$, which corresponds to the factor by which we divided.

EXERCISES

Solve by factoring and check:

1. $x^2 = 4$.
2. $x^2 = 16$.
3. $x^2 = 49$.
4. $x^2 = 5x$.
5. $x^2 + 12 = 7x$.
6. $x^2 - x = 90$.
7. $x^2 + x = 56$.
8. $6x^2 - 18x = 0$.
9. $8 + x^2 = 9x$.
10. $5x^2 = 25x$.
11. $x^2 - 11x = 26$.
12. $x^2 = 19x - 34$.
13. $2x^2 + 5x + 3 = 0$.
14. $3x^2 + 5x + 2 = 0$.
15. $3x^2 = 11x + 20$.
16. $10x^2 = 13x + 9$.
17. $10x^2 = 19x - 6$.
18. $(x + 7)(x - 2) = 10$.
19. $(x + 3)(2x - 1) - 15 = 0$.
20. $(x - 3)^2 + (x + 3)(x - 3) = 0$.
21. $(2x - 1)^2 + (2x + 5)(2x - 1) = 0$.
22. $(3x - 2)^2 - (3x - 2)(x - 7) = 0$.
23. $x^2 - 4 = 3x + 6$.

24. $x^2 - 3bx + 2b^2 = 0.$

25. $x^2 + ax - cx = 0.$

26. $x^2 - nx + cx - cn = 0.$

27. $x^2 - nx = 5x - 5n.$

28. $(x-a)^2 - (2x-1)(x-a) = 0.$

29. $(3x+2)(x-2n) - (x-2n)^2 = 0.$

30. $x^2 - 4 - 3x + 6 = 0.$

31. $x^2 - a^2 - cx + ac = 0.$

81. Cubic equations. An equation in x which may be put in the form

$$ax^3 + bx^2 + cx + d = 0,$$

where the coefficients a, b, c, and d represent numbers, is called a *cubic equation*, or an equation of the third degree.

Some equations of higher degree than the second may be solved by the method of factoring.

EXAMPLE

Solve the cubic equation $x^3 + x^2 = 4x + 4.$

Solution. Transposing, $x^3 + x^2 - 4x - 4 = 0.$

Factoring, $x^2(x+1) - 4(x+1) = (x+1)(x^2-4)$
$= (x+1)(x-2)(x+2).$

Setting each factor equal to zero,

$x + 1 = 0,$ therefore $x = -1;$
$x - 2 = 0,$ therefore $x = 2;$
$x + 2 = 0,$ therefore $x = -2.$

Check. When $x = -1,$ $-1 + 1 = -4 + 4.$
When $x = 2,$ $8 + 4 = 8 + 4.$
When $x = -2,$ $-8 + 4 = -8 + 4.$

EXERCISES

Solve and check the following:

1. $x^3 - 9x = 0$.
2. $x^3 = 4x$.
3. $x^3 - x^2 - 20x = 0$.
4. $x^3 = x^2 + 12x$.
5. $x^3 + 6x = 5x^2$.
6. $2y^3 - 32y = y^2 - 16$.
7. $y^3 - 9y = 45 - 5y^2$.
8. $x^3 + 2 = x + 2x^2$.
9. $y^3 + 4y^2 = 36 + 9y$.
10. $6x^2 - 9x = 54 - x^3$.
11. $y^4 - 5y^2 + 4 = 0$.
12. $y^4 + 9 = 10y^2$.
13. $y^4 + 36 = 13y^2$.
14. $x^3 - ax^2 = 4n^2x - 4an^2$.

PROBLEMS

1. The square of a certain number plus the number itself equals 30. Find the number.

HINT. Translated into an equation this becomes $n^2 + n = 30$.

2. Four times the square of a certain number equals nine times the number. What is the number?

HINT. Translated into an equation this becomes $4n^2 = 9n$.

3. If to the square of a certain number the sum of twice the number and 7 be added, the result is 70. Find the number.

4. If from the square of a certain number twice the number be taken, the result is 24. Find the number.

5. A certain number is added to 19 and the same number is added to 25. The product of the two sums is 720. Find the number.

6. A certain number is subtracted from 17 and is also subtracted from 31. The product of the remainders is 240. Find the number.

7. The difference between two numbers is 7 and the difference of their squares is 203. Find the numbers.

8. A certain number is added to 23 and subtracted from 32. The product of the two results obtained is 92 more than 14 times the number. Find the number.

9. If from the square of three times a certain number 20 times the number be taken, the result will be 16 times the number. Find the number.

10. The depth of a certain lot whose area is 4800 square feet is three times its frontage. Find the dimensions of the lot.

11. The area of the floor of a certain room is 80 square yards and the room is 6 feet longer than it is wide. Find the dimensions of the room.

12. The area of a rectangular field is 80 square rods. The field is 11 yards longer than it is wide. Find its dimensions.

13. The sum of the squares of two consecutive numbers is 181. Find the numbers.

14. The sum of the squares of two consecutive odd numbers is 130. Find the numbers.

15. The difference of the squares of two consecutive even numbers is 76. Find the numbers.

16. The sum of the squares of three consecutive odd numbers is 155. Find the numbers.

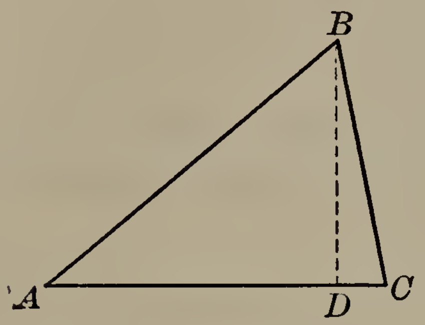
17. An uncovered square box 9 inches deep has 252 square inches of inside surface. Find its dimensions.

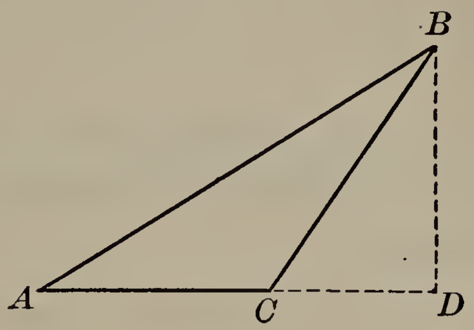
18. The entire outer surface of a cube is 150 square inches. Find the edge of the cube.

19. A rectangular box is four times as long and three times as wide as it is deep. There are 950 square inches in its entire outer surface. Find the dimensions.

20. A box is 5 inches longer and 3 inches wider than it is deep. There are 180 square inches in its entire outer surface. Find its dimensions.

The *altitude* of a triangle is the perpendicular from any vertex to the side opposite. This side is called the *base.*





In the adjacent figures, BD is the altitude and AC is the base of each triangle.

If a is the altitude of a triangle and b its base, the area of the triangle is $\frac{ab}{2}$.

In making use of this and similar formulas the unit in terms of which the lines are measured must be stated.

21. The area of a triangle is 42 square feet and the altitude is 7 feet. Find the base.

22. The altitude of a triangle is four times the base and the area is 50 square feet. Find the base and the altitude.

23. The base of a triangle is three times the altitude and the area is 54 square feet. Find the base and the altitude.

24. The area of a triangle is 144 square feet and the base is eight times the altitude. Find the base and the altitude.

25. The area of a triangle is 96 square feet and the base is 4 feet longer than the altitude. Find the base and the altitude.

HINT. Let $x =$ altitude in feet.

Then $x + 4 =$ base in feet,

and the area $= \frac{x(x+4)}{2} = 96$,

or $x^2 + 4x = 192$, etc.

26. The altitude of a triangle is 3 feet longer than the base. The area is 10 square yards. Find the base and the altitude.

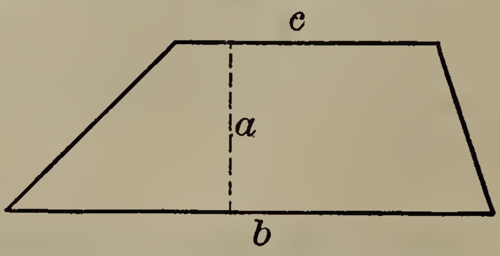
27. In a triangle the two sides about the right angle differ by 5 feet. The area of the triangle is 150 square feet. Find the sides about the right angle.

28. The area of a triangle is 81 square feet and the altitude is twice the base. Find the base and the altitude.

29. The area of a triangle is $6\frac{2}{3}$ square feet and the altitude is 28 inches longer than the base. Find the base and the altitude.

30. The sum of the two shorter sides of a right triangle is 34 feet and the area is 120 square feet. Find the base and the altitude of the triangle.

31. The area of a triangle is 12 square yards and the altitude is 3 feet longer than three times the base. Find the base and the altitude.



A *trapezoid* is a four-sided figure two of whose sides are unequal and parallel.

The bases of a trapezoid are the two parallel sides, b and c.

The altitude, a, is the perpendicular distance between the bases.

The area of a trapezoid is given by the formula $\frac{a(b+c)}{2}$.

32. Find the area of a trapezoid whose bases are 12 feet and 20 feet respectively, and whose altitude is 9 feet.

33. The altitude of a trapezoid is 10 inches, its area is 135 square inches, and one base is 5 inches longer than the other. Find the bases.

HINT. Let $x =$ the length of one base in inches.

Then $x + 5 =$ the length of the other base in inches,

and the area $= \frac{(x + x + 5)10}{2} = 135$,

or $20x + 50 = 270$, etc.

34. One base of a trapezoid is 10 feet, the other base is three times the altitude, and the area is 84 square feet. Find the altitude and the other base.

35. The altitude of a trapezoid is one half the shorter base, and the latter is two thirds of the other base. The area is 160 square feet. Find the bases and the altitude.

36. The bases of a trapezoid are respectively 4 feet and 8 feet longer than the altitude, and the area is 352 square feet. Find the bases and the altitude.

37. One base of a trapezoid is 14 feet longer than the other, and the altitude is one third the sum of the bases. The area is 54 square yards. Find the bases and the altitude.

38. The area of a trapezoid is 20 square yards, the altitude equals one base, and the other base exceeds the altitude by 6 feet. Find the bases and the altitude.

39. One base of a trapezoid exceeds the other by 12 feet, the altitude is 3 feet longer than five times the shorter base, and the area is 44 square yards. Find the bases.

REVIEW EXERCISES

1. Does $x = 7$ satisfy the equation $2x^2 + x - 66 = 0$? Does $x = -6$ satisfy it?

2. Give an example of (*a*) a linear equation, (*b*) a quadratic equation, (*c*) a cubic equation.

3. Is zero a root of the equation $x^3 + 5 - 4x = 5$? Is 2 a root? Is -2 a root?

4. Give an example of an equation of the second degree. Give one of the third degree.

5. Is 4 a root of the equation $\frac{30}{x-2} - 2x = 4$? Is 3 a root?

6. What conclusion can be drawn from the statement $25(7x - 35) = 0$? Explain.

7. Solve $2x^2 + 5x + 3 = 0$.

8. Solve $2x^2 + 5x = 88$.

9. Solve for x the equation $x^2 - 2ax + abx = 2a^2b$.

10. Given $t^3 - 64t = 0$ and $t^2 - 64 = 0$. If we solve the second equation, have we solved the first? Explain the point illustrated by these two equations.

11. If $abc = 0$, what conclusion regarding the values of a, b, or c can be drawn? What possible values may exist for a, b, and c?

12. The product of two numbers whose difference is 4 equals 357. Find the numbers.

13. Four times the square of a certain number minus four times the number equals 399. Find the number.

14. A verbal problem apparently easily solved may present an impossible situation and lead to impossible results.

Note the following: A double-decked bus carried 25 passengers. Two more passengers rode inside than on top. Find the number of passengers riding inside.

15. Nine times the square of a certain number of books minus 15 times that number is 176. How many books are there? Is this result possible?

16. In Exercise 15 change the word "books" to "days." Are the results possible? Change it to "motor cars." Are the results possible?

17. The area of a square in square yards and its perimeter in feet are expressed by the same number. Find the dimensions.

18. Find two consecutive integers whose product is 210.

19. The product of a certain even number and the second odd number greater in value is 550. Find the odd and the even number.

20. The product of a certain odd number and the second greater even number is 208. Find the two numbers.

CHAPTER XVI

FRACTIONS

82. Algebraic fractions. In arithmetic the student learned the properties of such fractions as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{12}$, .2, .36, etc., and how to perform the four fundamental operations on such fractions. We must now study the properties of algebraic fractions and learn to perform the four fundamental operations on them.

The expression $\frac{a}{b}$, in which a and b represent numbers or polynomials, is an *algebraic* fraction. It is read "a divided by b," or "a over b." A fraction is an indicated quotient in which the dividend is the numerator and the divisor the denominator. The numerator and denominator are often called the *terms* of a fraction.

Certain operations upon fractions, such as multiplying both numerator and denominator by a number (raising to higher terms), and dividing both numerator and denominator by a number (reducing to lower terms), are often necessary before the processes of addition or subtraction of two or more fractions can be performed.

The change of a fraction to lower or to higher terms, and the addition and the subtraction of fractions in both arithmetic and algebra, depend on the

PRINCIPLE. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus $\frac{4}{5}=\frac{4\cdot 3}{5\cdot 3}=\frac{12}{15}$ and $\frac{24}{42}=\frac{24\div 6}{42\div 6}=\frac{4}{7}$.

Similarly, $\frac{x}{z}=\frac{x\cdot a}{z\cdot a}=\frac{ax}{az}$ and $\frac{x\div n}{z\div n}=\frac{x/n}{z/n}$.

Since $\frac{3}{3}$, $\frac{6}{6}$, $\frac{a}{a}$, and $\frac{n}{n}$ are each equal to 1, each of the four preceding illustrations is really a multiplication or a division of a fraction by 1. This produces no change in the numerical value of *any* fraction, though it may change its form.

ORAL EXERCISES

What is the fraction obtained by multiplying the numerator and the denominator of each of the following fractions by the number on the right?

1. $\frac{1}{2}$, 3.
2. $\frac{2}{3}$, 4.
3. $\frac{2}{5}$, 3.
4. $\frac{3}{7}$, 4.
5. $\frac{3}{x}$, 5.
6. $\frac{a}{n}$, c.
7. $\frac{3}{n+2}$, a.
8. $\frac{a-n}{a+n}$, $2n$.

How is the second fraction obtained from the first?

9. $\frac{1}{3}$, $\frac{3}{9}$.
10. $\frac{1}{4}$, $\frac{2}{8}$.
11. $\frac{3}{5}$, $\frac{6}{10}$.
12. $\frac{2}{7}$, $\frac{6}{21}$.
13. $\frac{5}{9}$, $\frac{10a}{18a}$.
14. $\frac{7}{8x}$, $\frac{14a}{16ax}$.

What fractions are obtained by dividing both the numerator and the denominator of each of the following fractions by the number on the right?

15. $\frac{3}{6}$, 3.
16. $\frac{6}{15}$, 3.
17. $\frac{12}{30}$, 6.
18. $\frac{15}{25}$, 5.
19. $\frac{6n}{9}$, 3.
20. $\frac{n}{nx}$, n.
21. $\frac{3n}{n+nx}$, n.
22. $\frac{n+an}{n-an}$, n.

How is the second fraction obtained from the first?

23. $\frac{6}{9}, \frac{2}{3}$.

24. $\frac{12}{21}, \frac{4}{7}$.

25. $\frac{24}{32}, \frac{3}{4}$.

26. $\frac{n^2}{nx}, \frac{n}{x}$.

27. $\frac{nx^2}{n^2x}, \frac{x}{n}$.

28. $\frac{3n^3}{9n^2y}, \frac{n}{3y}$.

83. Reduction of fractions to lowest terms. A fraction is in its *lowest terms* when no factor except 1 is common to both numerator and denominator.

Cancellation is the process of dividing the numerator and the denominator of a fraction by a factor common to both.

EXAMPLES

Reduce to lowest terms:

1. $\frac{63a^3b^2c^2}{84ab^2c^4}$.

Solution. $$\frac{63a^3b^2c^2}{84ab^2c^4} = \frac{\overset{3}{\cancel{3^2}} \cdot \cancel{7} \cdot \overset{a^2}{\cancel{a^3}}\cancel{b^2}\cancel{c^2}}{2^2 \cdot \cancel{3} \cdot \cancel{7}\,\cancel{a}\cancel{b^2}\underset{c^2}{\cancel{c^4}}} = \frac{3a^2}{4c^2}.$$

2. $$\frac{4a^2n - 36n}{3a^2n - 18an + 27n} = \frac{4\cancel{n}(a+3)\cancel{(a-3)}}{3\cancel{n}(a-3)\cancel{(a-3)}} = \frac{4a+12}{3a-9}.$$

The pupil should note that a factor which occurs one or more times in both numerator and denominator of a fraction can be canceled only the same number of times from each.

For reducing a fraction to its lowest terms we have the

Rule. *Separate the numerator and the denominator into their prime factors and cancel the factors common to both.*

Cancellation as used in the rule means an actual division of the numerator and the denominator by the same expression. Therefore *only factors which are common to the numerator and the denominator can be canceled.*

The terms (the parts connected by plus or minus signs) in polynomial numerators and denominators, even if alike, can never be canceled. For example, it would be incorrect to "cancel" thus: $\frac{7+\cancel{3}}{11+\cancel{3}}$, as the resulting fraction would be $\frac{7}{11}$ instead of the true value, $\frac{10}{14}$, or $\frac{5}{7}$. Similarly, in the fraction $\frac{a+z+3t^2}{x+z+6t^2}$ no cancellation is possible.

We have seen that we may multiply or divide both numerator and denominator of a fraction by the same number without affecting the value of the fraction. But we should never forget that *adding the same number to or subtracting the same number from both numerator and denominator changes the value of the fraction. Also, squaring both numerator and denominator leads to a different value.* Compare this statement with the operations that may be performed on each member of an equation as given on pages 56–58.

ORAL EXERCISES

Reduce to lowest terms:

1. $\frac{18}{24}$.
2. $\frac{27}{36}$.
3. $\frac{12}{42}$.
4. $\frac{3ax}{15a^2}$.
5. $\frac{10x^2}{25x^3}$.
6. $\frac{14a}{42a^2}$.
7. $\frac{8x}{56x^3}$.
8. $\frac{9a^2}{54a}$.
9. $\frac{60m^2}{90m^3}$.
10. $\frac{36a}{84ax}$.
11. $\frac{18ax}{72x^2}$.
12. $\frac{24ax}{60a^2x}$.
13. $\frac{36a}{96a^2}$.
14. $\frac{121x}{77x^2}$.
15. $\frac{144x^2}{120x}$.
16. $\frac{108am}{144a^2m}$.

17. $\frac{66\,xn^2}{110\,an}$.

18. $\frac{55\,az}{132\,a^2z}$.

19. $\frac{64\,m^2}{128\,m^3}$.

20. $\frac{132\,x^4}{144\,x^2}$.

21. $\frac{-13}{39}$.

22. $\frac{-12}{18}$.

23. $\frac{-14}{-21}$.

24. $\frac{3\,a^2x}{-36\,ax^2}$.

25. $\frac{-14\,m^3n^2}{120\,m^3n^3}$.

26. $\frac{-363\,x^5y^2}{-33\,x^3y^3}$.

27. $\frac{-45\,h^3k^2}{30\,hk^2}$.

28. $-\frac{160\,s^2t^3}{25\,st^4u}$.

EXERCISES

Reduce to lowest terms:

1. $\frac{32}{72}$.

2. $\frac{54}{81}$.

3. $\frac{a^3b}{a^2b^4}$.

4. $\frac{x^3n}{xn^3}$.

5. $\frac{an^3}{3\,a^2n}$.

6. $\frac{10\,an^2x}{5\,ax^2}$.

7. $\frac{15\,an^3}{25\,a^3n^2x}$.

8. $\frac{12\,x^2n^6}{18\,xn^7}$.

9. $\frac{42\,a^4n^4c}{56\,a^3n^6c^2}$.

10. $\frac{3\,a}{3\,a+3}$.

11. $\frac{4\,n-6\,n^2}{2\,n^3}$.

12. $\frac{3\,nx+3\,n^2}{3\,n^4}$.

13. $\frac{14\,n}{21\,n^2+14\,n}$.

14. $\frac{3\,n-2}{6\,n^2-4\,n}$.

15. $\frac{x^2-4}{(x-2)^2}$.

16. $\frac{n^2-2\,n+1}{n^2-1}$.

17. $\frac{n^3-4\,n}{(n+2)^2}$.

18. $\frac{n^2+3\,n-10}{n^2+4\,n-5}$.

19. $\frac{x^2+7\,x-18}{x^2-3\,x+2}$.

20. $\frac{(x^2-n^2)(x+3\,n)}{(x+n)^2(2\,x+3\,n)}$.

21. $\frac{n^2-25}{n^2-n-30}$.

22. $\frac{x^2-5\,xy+4\,y^2}{x^2-16\,y^2}$.

23. $\frac{3\,n^2-3\,n-270}{3\,n^2-243}$.

24. $\frac{x^4-16}{x^3-2\,x^2+4\,x-8}$.

25. $\frac{n^4-a^4}{n^4+3\,n^2a^2+2\,a^4}$.

26. $\frac{x^2-5\,x+6}{2\,x^2-12\,x+18}$.

27. $\frac{3a^2 - 6an + 3n^2}{a^2 + 6an - 7n^2}$.

28. $\frac{(x - 3)^2 - a^2}{2x - 6 + 2a}$.

29. $\frac{a^2 - (b - c)^2}{an + bn - cn}$.

30. $\frac{4n^3 - 5n^2 - 4n + 5}{8n^4 - 10n^3 + 12n - 15}$.

Supply the missing terms in the following expressions:

31. $\frac{3}{4} = \frac{?}{12}$.

32. $\frac{15}{16} = \frac{?}{48}$.

33. $\frac{9}{10} = \frac{18}{?}$.

34. $\frac{12}{32} = \frac{6}{?} = \frac{?}{8}$.

Change the following to equivalent fractions having as denominators the expressions given at the right:

35. $\frac{12n}{m}$, $12m^2n$.

36. $\frac{ab}{c}$, abc.

37. $\frac{m + n}{m - n}$, $m^2 - n^2$.

38. $\frac{5a + b}{3a + 2b}$, $15a^2 + 13ab + 2b^2$.

84. Lowest common multiple. The *lowest common multiple* (L.C.M.) of two or more arithmetical or algebraic expressions is the expression having the least number of factors which will exactly contain each of the given expressions.

If two or more polynomials have no common factor other than 1, they are said to be *prime* to each other.

The L.C.M. of two prime expressions is their product.

Thus $7x^2z$ and $11at^2$ are prime to each other, as also are $3x^2 - 5x$ and $x^2 - 16$. But $a^2 - 16$ and $a^2 - 5a + 4$ are not prime to each other, since each contains the factor $a-4$.

ORAL EXERCISES

Determine by inspection the L.C.M. of:

1. 4, 10.
2. 6, 8.
3. 6, 9.
4. 6, 10.
5. 4, 10.
6. 10, 15.
7. 10, 25.
8. 4, 14.
9. 7, 14.
10. 4, 12.
11. 6, 12.
12. 8, 12.

13. 10, 12.

14. 2, 4, 6.

15. 3, 6, 9.

16. 2, 5, 15.

17. 3, 5, 10.

18. 6, 5, 10.

19. 4, 6, 8.

20. 6, 8, 12.

21. 4, 9, 12.

22. Give one other common multiple for the numbers in Exercises 1–10.

EXAMPLE

Find the L.C.M. of $18\,x^2y$, $15\,xy^2$, $20\,x^3y^4$.

Solution.

$$\begin{aligned} 18\,x^2y &= 2\cdot 3^2\cdot x^2\cdot y,\\ 15\,xy^2 &= 3\cdot 5\cdot x\cdot y^2,\\ 20\,x^3y^4 &= 2^2\cdot 5\cdot x^3\cdot y^4. \end{aligned}$$

Since the L.C.M. must contain each of the expressions, it must contain as factors 2^2, 3^2, 5, x^3, and y^4. If it does this it will contain the other factors, 2, 3, 5, x^2, y, x, and y^2, and hence will contain any of the above numbers.

Therefore the L.C.M. $= 2^2\cdot 3^2\cdot 5\cdot x^3\cdot y^4 = 180\,x^3y^4$.

The method of finding the L.C.M. of two or more expressions is stated in the following

RULE. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

EXERCISES

Find the L.C.M. of:

1. 18, 24, 36.

2. 25, 30, 35.

3. 24, 36, 44.

4. 54, 75, 96.

5. n^2x, n^2x^2, nx^2.

6. an, a^2, n^2.

7. n^3x, nx^3, n^2x^2.

8. $4\,n^2$, $6\,nx^2$, $9\,n^3x$.

9. $9\,n^2$, $6\,n^3$, $12\,n^5$.

10. $3ax$, $5a^2x$, $7a$.

11. $3a$, $4b$, $6nx$.

12. $5x^2y$, $10y^2z$, $4x^2yz^2$.

13. $36ax^2$, $42axy$, $63x^2y$.

14. $4n$, $n^2 - bn$.

15. $12an$, $3an^2 - 3a^2n$.

16. $a^2 - an$, a^2n.

17. nx^2, $3n^2x$, $9an^2 - 6a^2n$.

18. $cx + cy$, $dx + dy$.

19. $3n + 3x$, $6an + 6ax$.

20. $n^2 - nx$, $3an - 3ax$.

21. $ax^2 - 9a$, $x^2 - 5x + 6$, $x^2 - 4x + 4$.

Solution.

$$ax^2 - 9a = a(x-3)(x+3).$$
$$x^2 - 5x + 6 = (x-2)(x-3).$$
$$x^2 - 4x + 4 = (x-2)(x-2).$$

Therefore the L.C.M. $= a(x-3)(x+3)(x-2)^2$.

22. $x^2 - 4$, $x^2 - x - 6$.

23. $x^2 - ax$, $x^2 - a^2$.

24. $n^2 - 4$, $n^2 - 8n - 20$.

25. $2n^3 - 2n$, $3n^4 + 15n^3 - 18n^2$.

26. $a^2 + ab - 2b^2$, $ac - 2bd + 2ad - bc$.

27. $a^3 - a$, $a^3 - a^2 - 2a$, $a^2 - 2a$.

85. Equivalent fractions. Two fractions are equivalent when one can be obtained from the other either by multiplying or by dividing both numerator and denominator by the same expression.

For example, $\frac{2}{3}$ and $\frac{10}{15}$ are equivalent fractions; also $\frac{nx}{n^2}$ and $\frac{x}{n}$.

The *lowest common denominator* (L.C.D.) of two or more fractions is the L.C.M. of their denominators.

Before adding or subtracting fractions it is necessary to find equivalent fractions having the L.C.D.

EXAMPLES

Reduce to respectively equivalent fractions having the lowest common denominator:

1. $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{9}$.

Solution. The L.C.M. of the denominators is 36. Multiplying the numerator and denominator of the first fraction by 12, of the second by 9, and of the third by 4, we obtain $\frac{24}{36}$, $\frac{27}{36}$, and $\frac{20}{36}$ respectively.

2. $\frac{3a}{4n^2x}$ and $\frac{5n}{6ax^2}$.

Solution. The L.C.M. of the denominators is $12an^2x^2$. By inspection it is seen that $4n^2x$ multiplied by $3ax$ gives $12an^2x^2$, and $6ax^2$ multiplied by $2n^2$ gives $12an^2x^2$. Multiplying the numerator and the denominator of the first fraction by $3ax$ and of the second by $2n^2$ gives $\frac{9a^2x}{12an^2x^2}$ and $\frac{10n^3}{12an^2x^2}$ respectively.

Thus,
$$\frac{3a}{4n^2x} = \frac{3a \cdot 3ax}{4n^2x \cdot 3ax} = \frac{9a^2x}{12an^2x^2}$$

and
$$\frac{5n}{6ax^2} = \frac{5n \cdot 2n^2}{6ax^2 \cdot 2n^2} = \frac{10n^3}{12an^2x^2}.$$

Therefore, to change two or more fractions (in their lowest terms) to respectively equivalent fractions we have the

RULE. *Write the fractions with their denominators in factored form if they are not already so expressed.*

Find the L.C.M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L.C.M. which are not found in the denominator of that fraction.

ORAL EXERCISES

Transform to respectively equivalent fractions having the lowest common denominator:

1. $\frac{1}{2}, \frac{1}{4}$.
2. $\frac{1}{6}, \frac{1}{3}$.
3. $\frac{1}{2}, \frac{1}{6}$.
4. $\frac{1}{3}, \frac{1}{4}$.
5. $\frac{1}{4}, \frac{1}{16}$.
6. $\frac{1}{8}, \frac{1}{12}$.
7. $\frac{1}{6}, \frac{1}{9}$.
8. $\frac{1}{3}, \frac{1}{5}$.
9. $\frac{1}{4}, \frac{1}{5}$.
10. $\frac{2}{3}, \frac{1}{2}$.
11. $\frac{2}{3}, \frac{3}{5}$.
12. $\frac{1}{6}, \frac{1}{8}$.
13. $\frac{1}{9}, \frac{1}{12}$.
14. $\frac{1}{10}, \frac{1}{15}$.
15. $\frac{3}{10}, \frac{2}{5}$.
16. $\frac{3}{5}, \frac{4}{9}$.
17. $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$.
18. $\frac{1}{3}, \frac{1}{2}, \frac{1}{9}$.
19. $\frac{1}{4}, \frac{1}{6}, \frac{1}{12}$.
20. $\frac{2}{3}, \frac{3}{4}, \frac{1}{5}$.
21. $\frac{1}{a}, \frac{1}{a^2}$.
22. $\frac{1}{ax}, \frac{1}{cx}$.
23. $\frac{2}{x^2}, \frac{3}{x}$.
24. $\frac{3}{n^2}, \frac{2}{n^3}$.
25. $\frac{1}{a}, \frac{1}{b}$.
26. $\frac{2}{a}, \frac{3}{ab}$.
27. $\frac{4}{an}, \frac{5}{n^2}$.
28. $\frac{1}{b}, \frac{c}{d}$.
29. $\frac{a}{b}, \frac{c}{d}$.

EXERCISES

Transform to respectively equivalent fractions having the lowest common denominator:

1. $\frac{a}{2}, \frac{x}{3}$.
2. $\frac{2x}{3}, \frac{3a}{2}$.
3. $\frac{2n}{5}, \frac{3an}{4}, \frac{a^2}{6}$.
4. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.
5. $\frac{a}{x}, \frac{x}{a}$.
6. $\frac{a}{n}, \frac{t}{x}$.
7. $\frac{a}{b}, \frac{c}{d}, \frac{x}{y}$.
8. $\frac{a}{x^2}, \frac{c}{x}, \frac{t}{3}$.
9. $\frac{3a}{2b^2c}, \frac{2b}{3a^2c}$, and $\frac{c}{6ab}$.
10. $\frac{3y}{2n^3z}, \frac{9x}{4ntz}$, and $\frac{7xy}{5nt^2z}$.
11. $\frac{a+b}{c}, \frac{a-b}{a}$.
12. $\frac{b-c}{2a}, \frac{a+b}{3c}$.
13. $\frac{2+x^2}{2x}, \frac{4+x^2}{4n}$.
14. $\frac{a-x}{mn}, \frac{am-x}{n^2}$.

15. $\frac{2}{a+3}, \frac{5}{a-3}$.

Solution. By inspection it is seen that the L. C. D. is the product of the two denominators, or $(a+3)(a-3)$. Hence the multiplier for the numerator and denominator of the first fraction is $a-3$, and for the second is $a+3$. Using these, we obtain

$$\frac{2}{a+3}=\frac{2a-6}{a^2-9} \text{ and } \frac{5}{a-3}=\frac{5a+15}{a^2-9}.$$

16. $\frac{1}{a+2}, \frac{2}{a-2}$.

18. $\frac{a}{a+n}, \frac{n}{a-n}$.

17. $\frac{x}{x-3}, \frac{2x}{x+3}$.

19. $\frac{x+2}{x-2}, \frac{x-2}{x+2}$.

20. $\frac{x+1}{x-2}, \frac{x-2}{x+1}$.

21. $\frac{a}{x+2}, \frac{3}{x-2}, \frac{a}{x}$.

HINT. The L. C. D. $= x(x-2)(x+2)$. Hence the numerator and denominator of the first fraction must be multiplied by $x(x-2)$, of the second by $x(x+2)$, and of the third by $(x+2)(x-2)$.

22. $\frac{1}{a}, \frac{2}{b}, \frac{3}{a+b}$.

27. $\frac{3x}{x^2-4}, \frac{5x}{6+3x}, \frac{x}{x-2}$.

23. $\frac{a}{x}, \frac{b}{ax}, \frac{c}{a+x}$.

28. $\frac{2x}{x-3}, \frac{5}{x^2-9}, \frac{5}{x^2-5x+6}$.

24. $\frac{2}{x+3}, \frac{3}{2x+6}$.

29. $\frac{x+3}{x^2-25}, \frac{x+1}{x^2-6x+5}, \frac{2}{x+5}$.

25. $\frac{n}{ax+a^2}, \frac{2}{x+a}$.

30. $\frac{1}{4a^2-1}, \frac{2a-1}{2a+1}, \frac{3}{a}$.

26. $\frac{a}{x^2-n^2}, \frac{b}{x+n}$.

31. $\frac{x+1}{2x+3}, \frac{3x}{2x^2+5x+3}$.

NOTE. The problem of operating with fractions presented great difficulties to all the early races. The Egyptians and the Greeks, even down to the sixth century of our era, always reduced their fractions to the sum of several fractions each of which had 1 for a numerator. For example, $\frac{5}{8}$ would be expressed as $\frac{1}{2}+\frac{1}{8}$. The Romans usually expressed all the fractions of a sum in terms of fractions with the common denominator 12. The Babylonians resorted to a similar device, but used 60 for the denominator. In some way they all attempted to evade the difficulty of considering changes in both numerator and denominator. The Hindus seem to have been the first to reduce fractions to a common denominator, though Euclid (300 B.C.) was familiar with the method of finding the least common multiple of two or more numbers.

86. Addition and subtraction of fractions. If two or more fractions have the same denominator, their sum is the fraction obtained by adding their numerators and writing the result over their common denominator.

For example, $\frac{3}{7}+\frac{2}{7}+\frac{6}{7}=\frac{11}{7}$, and $\frac{n}{d}+\frac{3n}{d}+\frac{5n}{d}=\frac{9n}{d}$.

If two fractions have the same denominators, their difference is the fraction obtained by subtracting the numerator of the subtrahend from the numerator of the minuend and writing the result over their common denominator.

For example, $\frac{6}{11}-\frac{2}{11}=\frac{4}{11}$, and $\frac{a}{x}-\frac{c}{x}=\frac{a-c}{x}$.

If it is required to add or to subtract two fractions having unlike denominators, the fractions must be changed to respectively equivalent fractions having a common denominator; then their sum or their difference is obtained as explained above.

For example, to find the sum of $\frac{2}{3}+\frac{3}{5}+\frac{5}{6}$ we reduce the fractions to respectively equivalent fractions having the common denominator 30, by multiplying both numerator and denominator of $\frac{2}{3}$ by 10, of $\frac{3}{5}$ by 6, and of $\frac{5}{6}$ by 5. The fractions become $\frac{20}{30}$, $\frac{18}{30}$, and $\frac{25}{30}$ respectively, and their sum is $\frac{63}{30}$, or $\frac{21}{10}$.

The pupil should always reduce fractional results to their lowest terms.

ORAL EXERCISES

Find the algebraic sum of:

1. $\frac{1}{3}+\frac{4}{3}$.
2. $\frac{4}{5}+\frac{6}{5}$.
3. $\frac{4}{7}+\frac{5}{7}$.
4. $\frac{6}{11}-\frac{2}{11}$.
5. $\frac{5}{12}-\frac{3}{12}$.
6. $\frac{11}{18}+\frac{5}{18}$.
7. $\frac{1}{n}+\frac{2}{n}$.
8. $\frac{3}{n}+\frac{a}{n}$.
9. $\frac{10}{a}-\frac{3}{a}$.
10. $\frac{n}{ax}-\frac{3}{ax}$.
11. $\frac{n+1}{a}-\frac{2}{a}$.
12. $\frac{a-3}{x}+\frac{5}{x}$.
13. $\frac{a+c}{ax}-\frac{c}{ax}$.
14. $\frac{a-n}{3x}-\frac{a+n}{3x}$.
15. $\frac{a^2+n^2}{a+n}-\frac{n^2-a^2}{a+n}$.
16. $\frac{a^2-ax}{a+x}+\frac{2ax}{a+x}$.
17. $\frac{n^3-n^2}{n+1}+\frac{2n^2}{n+1}$.
18. $\frac{2a+n}{a+an}-\frac{2n+a}{a+an}$.

In adding or subtracting algebraic fractions with unlike denominators, as $\frac{a}{x}$ and $\frac{c}{z}$, we proceed in a similar way, as follows:

Multiply both terms of $\frac{a}{x}$ by z, and of $\frac{c}{z}$ by x. The fractions become $\frac{az}{xz}$ and $\frac{cx}{xz}$ respectively, the sum of which is $\frac{az+cx}{xz}$. Similarly, $\frac{x}{y}-\frac{n}{d}=\frac{xd}{yd}-\frac{ny}{yd}$, which equals $\frac{xd-ny}{yd}$.

EXAMPLE

Find the algebraic sum of $\frac{3x}{5n^2} + \frac{x-5}{15a} - \frac{2x-n}{3an}$.

Solution. The L. C. D. is $15an^2$.

$$\frac{3x}{5n^2} + \frac{x-5}{15a} - \frac{2x-n}{3an} = \frac{3x \cdot 3a}{5n^2 \cdot 3a} + \frac{(x-5)n^2}{15a \cdot n^2} - \frac{(2x-n)5n}{3an \cdot 5n}$$

$$= \frac{9ax}{15an^2} + \frac{n^2x - 5n^2}{15an^2} - \frac{10nx - 5n^2}{15an^2}$$

$$= \frac{9ax + (n^2x - 5n^2) - (10nx - 5n^2)}{15an^2}$$

$$= \frac{9ax + n^2x - 5n^2 - 10nx + 5n^2}{15an^2}$$

$$= \frac{9ax + n^2x - 10nx}{15an^2}.$$

Check. The above solution gives:

$$\frac{3x}{5n^2} + \frac{x-5}{15a} - \frac{2x-n}{3an} = \frac{9ax + n^2x - 10nx}{15an^2}.$$

Now let $x = 1$, $a = 2$, and $n = 3$, and we obtain

$$\frac{3}{45} + \frac{1-5}{30} - \frac{2-3}{18} = \frac{18+9-30}{270}.$$

$$\frac{6-12+5}{90} = \frac{-3}{270}, \text{ or } \frac{-1}{90} = \frac{-1}{90}.$$

Therefore, to find the algebraic sum of two or more fractions (in their lowest terms) we have the

RULE. *Reduce the fractions to respectively equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each numerator in a parenthesis preceded by the sign of the corresponding fraction.*

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator and, if necessary, reduce the resulting fraction to its lowest terms.

EXERCISES

Find the algebraic sum of:

1. $\frac{2}{3}+\frac{3}{5}$. **3.** $\frac{7}{16}+\frac{5}{12}$. **5.** $\frac{4}{5}+\frac{5}{11}$. **7.** $\frac{5}{26}+\frac{8}{39}-\frac{17}{52}$.
2. $\frac{5}{6}+\frac{3}{8}$. **4.** $\frac{5}{21}+\frac{3}{14}$. **6.** $\frac{3}{4}+\frac{1}{3}+\frac{7}{15}$. **8.** $\frac{5}{6}+\frac{7}{12}-\frac{4}{9}$.

9. $\frac{5n}{3}+\frac{6n}{5}$.

10. $\frac{3n}{5}+\frac{2n}{3}+\frac{4n}{15}$.

11. $\frac{x+3}{6}-\frac{3x+5}{8}$.

12. $\frac{5n-4}{12}-\frac{2n+8}{15}$.

13. $\frac{3x-2}{4}-\frac{5-x}{6}+\frac{5x}{9}$.

14. $\frac{1}{a}+\frac{1}{b}$.

15. $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$.

16. $\frac{a}{x}-\frac{x}{a}+\frac{2}{ax}$.

17. $\frac{2}{3x}+\frac{5}{2x^2}-\frac{10x}{4x^3}$.

18. $\frac{a+b}{a}-\frac{2b}{ab}-\frac{a}{b}$.

19. $\frac{a}{b}+\frac{b}{c}-\frac{c}{a}$.

20. $\frac{3}{n^2}+\frac{4}{n}-\frac{8}{3n^3}$.

21. $\frac{7}{a^2}-\frac{51}{ax}+\frac{6}{x^2}$.

22. $\frac{3}{ab}-\frac{5}{2a^2}-\frac{6}{5ab^3}$.

23. $\frac{m}{n}-\frac{5}{2n^3}+\frac{x}{4}$.

24. $\frac{5n^3-8}{9n^4}-\frac{7-2n^2}{4n^3}+\frac{3n-1}{6n^2}$.

25. $\frac{n^2-4}{nx^2}-\frac{4n^2-9}{2nx}+\frac{6-n}{5n^2}$.

26. $\frac{3t-4}{5t^3}-\frac{5-3t}{2t}+\frac{4t^2-5}{3t^2}$.

27. $\frac{2n}{5n^2x}-\frac{3x-2}{10nx^2}-\frac{4nx+5}{15n^2x^2}$.

28. $\dfrac{x+4}{x^2-9}-\dfrac{2x-3}{x^2-5x+6}.$

Solution.

$$\begin{aligned}\frac{x+4}{x^2-9}-\frac{2x-3}{x^2-5x+6}&=\frac{x+4}{(x-3)(x+3)}-\frac{2x-3}{(x-3)(x-2)}\\&=\frac{(x+4)(x-2)}{(x-3)(x+3)(x-2)}-\frac{(2x-3)(x+3)}{(x-3)(x-2)(x+3)}\\&=\frac{x^2+2x-8-(2x^2+3x-9)}{(x-3)(x-2)(x+3)}\\&=\frac{x^2+2x-8-2x^2-3x+9}{(x-3)(x-2)(x+3)}\\&=\frac{-x^2-x+1}{(x-3)(x-2)(x+3)}=\frac{1-x-x^2}{x^3-2x^2-9x+18}.\end{aligned}$$

(Unless otherwise directed the denominator should be retained in factored form throughout the solution.)

Check. Let $x=4$.

$$\frac{x+4}{x^2-9}-\frac{2x-3}{x^2-5x+6}=\frac{1-x-x^2}{x^3-2x^2-9x+18}.$$

$$\frac{4+4}{16-9}-\frac{8-3}{16-20+6}=\frac{1-4-16}{64-32-36+18}.$$

$$\frac{8}{7}-\frac{5}{2}=-\frac{19}{14},\quad\text{or}\quad\frac{16}{14}-\frac{35}{14}=-\frac{19}{14}.$$

In checking work in fractions we must assign such values to the letters as will make no denominator zero. This is necessary to avoid division by zero (see page 115). For this reason x in the above check cannot be 2, 3, or -3.

29. $\dfrac{3}{x^2-4}-\dfrac{2}{x^2-3x+2}.$

30. $\dfrac{3+n}{n-3}-\dfrac{4}{5}.$

31. $\dfrac{2}{x-7}+\dfrac{3}{x+7}.$

32. $\dfrac{4n}{n^2+nx}-\dfrac{11}{n}.$

33. $\frac{3x-y}{x^2-y^2}+\frac{5}{x-y}$.

34. $\frac{9}{x^2-16}+\frac{5}{x^2-9x+20}$.

35. $\frac{3x}{a^2-ax}-\frac{a+2x}{a^2-x^2}$.

36. $\frac{3n+2}{n+3}-\frac{5}{2n}+\frac{3}{4}$.

37. $\frac{8}{n^2-25}-\frac{1}{n^2-16n+55}$.

38. $\frac{n+2}{n^2-25}-\frac{2n-1}{n^2-6n+5}$.

39. $\frac{a-3}{a^2-6a}-\frac{2a+4}{a^2-8a+12}$.

40. $\frac{a^2-3ax+x^2}{a^2-6a+9}-\frac{3x-4a}{4a-12}$.

41. $\frac{n+3}{n^2+n}+\frac{2}{n}-\frac{5-n}{n^2+2n+1}$.

42. $\frac{a+x}{a^2-ax}-\frac{a-4x}{a^2-2ax+x^2}+\frac{3a-x}{a^2-x^2}$.

43. $\frac{2x+3}{x^2+3x-10}-\frac{3}{4x}+\frac{2-3x}{x^2+5x}$.

44. $\frac{x^2+3}{x^4-16}-\frac{2}{x^2-4}+\frac{3}{x^2+4}$.

45. $\frac{x-a}{x+a}-\frac{x+a}{x-a}-\frac{3a}{x}$.

87. Changes of sign in a fraction. The *sign of a fraction* is the plus or minus sign placed before the line separating the numerator from the denominator. Hence there are in a fraction three signs to consider: the sign of the fraction, the sign of the numerator, and the sign of the denominator.

Now in division the quotient of two expressions having like signs is positive, and the quotient of two expressions having unlike signs is negative.

Therefore

$$+\frac{+12}{+3}=+4; \qquad +\frac{-12}{-3}=+4;$$

$$-\frac{-12}{+3}=-(-4)=+4; \qquad -\frac{+12}{-3}=-(-4)=+4.$$

Or, in general terms, $+\frac{+a}{+b}=+\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}$.

These examples illustrate the

Principle. *Without altering the value of a fraction the following changes in sign may be made:*

(*a*) *The sign of the numerator and the sign of the denominator.*
(*b*) *The sign of the numerator and the sign before the fraction.*
(*c*) *The sign of the denominator and the sign before the fraction.*

Hence any fraction may be written in at least four ways, if proper changes of sign are made.

Thus, $\frac{4a}{2-3x}=\frac{-4a}{3x-2}=-\frac{-4a}{2-3x}=-\frac{4a}{3x-2}.$

Similarly,

$$\frac{a-2n}{3x-y+z}=\frac{2n-a}{-3x+y-z}$$

$$=-\frac{2n-a}{3x-y+z}=-\frac{a-2n}{-3x+y-z}.$$

The pupil should note particularly that changing the sign of the numerator involves a change of sign in each term of the numerator. Similarly, a change of sign of the denominator involves a change of sign in each term of the denominator.

Multiplying one factor of an indicated product by -1 changes the sign of every term of the expanded product.

Thus, $(n-3)(n-4)=n^2-7n+12.$

Multiplying the terms of the factor $n-3$ by -1, we have

$$(3-n)(n-4)=-n^2+7n-12.$$

Multiplying two factors of an indicated product by -1 does not change the sign of the expanded product.

As before, $(n-3)(n-4)=n^2-7n+12.$

But $(n-3)(-1)(n-4)(-1)=(3-n)(4-n)$

$$=n^2-7n+12.$$

From these illustrations it follows that changing the sign of an *odd* number of factors in an indicated product changes the sign of every term of the expanded product, but if the sign of an *even* number of factors is changed the sign of the expanded product is not changed.

EXERCISES

Write as equivalent fractions in three other ways:

1. $\frac{-n}{x}$.
2. $\frac{a}{-x}$.
3. $-\frac{t}{n}$.
4. $-\frac{3}{a-n}$.
5. $\frac{-2x}{a-b}$.
6. $\frac{3x-1}{2-5x}$.
7. $-\frac{a-b}{3x-2}$.
8. $\frac{a-x}{a^2-t^2}$.
9. $\frac{a-3}{x^2-2x-12}$.
10. $-\frac{3-x}{2x-5}$.

Write so that the letters are in alphabetical order in the factors of the denominator or so that the letter precedes the number if but one letter occurs in a factor:

11. $-\frac{3}{(3-x)(x+4)}$.
12. $-\frac{4}{(a+b)(b-a)}$.
13. $\frac{-5}{(x+5)(7-x)}$.
14. $\frac{-6}{(a-c)(c-a)}$.
15. $\frac{a-x}{(x-a)(x-c)}$.
16. $\frac{-x-3}{(5-x)(-3-x)}$.
17. $\frac{-7}{(a-b)(b-a)(c-a)}$.
18. $\frac{-8}{(3-a)(b-a)(c-a)}$.

Perform the indicated operation:

19. $\frac{x-3}{x-5}-\frac{3x}{5-x}$.

HINT. Rewrite so that the common denominator is $x-5$.

20. $\frac{n}{x-n}-\frac{x}{n-x}$.

21. $\frac{n-1}{n-a}+\frac{3n-2}{a-n}$.

22. $\frac{a-1}{2-n}+\frac{3-a}{n-2}+\frac{2a}{2-n}$.

23. $\frac{x}{3-x}-\frac{x^2+4x}{x^2-9}$.

HINT. $\frac{x}{3-x}-\frac{x^2+4x}{(x-3)(x+3)}=\frac{-x}{x-3}-\frac{x^2+4x}{(x-3)(x+3)}$ etc.

24. $\frac{3}{x^2-4}-\frac{5}{2-x}$.

25. $\frac{x+1}{5-x}-\frac{x-1}{x^2-25}$.

26. $\frac{x+4}{x^2-5x+6}-\frac{3}{3-x}+\frac{2}{2-x}$.

27. $\frac{x^2-tx}{x^2-4ax+3a^2}-\frac{x}{3a-x}+\frac{t}{a-x}$.

28. $\frac{t+3}{t^2-8t+15}-\frac{t}{5-t}-\frac{3}{3-t}$.

29. $\frac{2}{(a-b)(a-c)}-\frac{1}{(a-b)(c-a)}$.

30. $\frac{3n}{(n-3)(n-4)}-\frac{4n}{(3-n)(4-n)}$.

31. $\frac{6}{(a-7)(a-c)}-\frac{8}{(7-a)(c-a)}$.

32. $\frac{x-3}{x^2-5x+6}+\frac{2x-5}{9-x^2}$.

33. $\frac{3x-1}{x^3-9x}-\frac{3x-7}{x^2-6x+9}$.

34. $\frac{2a-3t}{a^2-5at+6t^2}-\frac{7t}{a^2-4t^2}$.

35. $\frac{x}{x^2-10x+24}-\frac{1}{6x-x^2-8}+\frac{3}{(6-x)(2-x)}$.

88. Reduction of a mixed expression to a fraction. The mixed number $4\frac{2}{3}$ really means $4+\frac{2}{3}$. It is equal to $\frac{4}{1}+\frac{2}{3}$, or $\frac{12}{3}+\frac{2}{3}, =\frac{14}{3}$. The corresponding case in algebra is $a+\frac{b}{c}$.

Now, $$a+\frac{b}{c}=\frac{a}{1}+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}.$$

Similarly,

$$n+2+\frac{3}{n-2}=\frac{n+2}{1}+\frac{3}{n-2}=\frac{(n+2)(n-2)}{n-2}+\frac{3}{n-2}$$
$$=\frac{n^2-4+3}{n-2}=\frac{n^2-1}{n-2}.$$

The process involved in this operation is nothing more than the addition of two fractions, one of which has the denominator 1.

ORAL EXERCISES

Reduce to fractional form:

1. $2+\frac{1}{3}$.
2. $3+\frac{1}{5}$.
3. $4+\frac{2}{3}$.
4. $5+\frac{3}{4}$.
5. $6+\frac{5}{7}$.
6. $8+\frac{3}{11}$.
7. $3+\frac{5}{12}$.
8. $4-\frac{5}{6}$.
9. $5-\frac{3}{5}$.
10. $6-\frac{2}{5}$.
11. $\frac{7}{5}-2$.
12. $\frac{14}{3}-3$.
13. $1+\frac{a}{x}$.
14. $2-\frac{t}{a}$.
15. $x-\frac{n}{x}$.
16. $n-\frac{3}{2x}$.
17. $x+\frac{x}{n}$.
18. $2a+\frac{3x}{2a}$.
19. $1-\left(\frac{1}{2}\right)^2$.
20. $a^2+\left(\frac{a}{2}\right)^2$.
21. $\frac{x^2}{4}+x$.
22. $x^2-\left(\frac{x}{2}\right)^2$.
23. $n^2-\left(\frac{n}{3}\right)^2$.
24. $3+\frac{2}{x+1}$.
25. $4-\frac{3}{n-2}$.
26. $x+1+\frac{1}{x-1}$.
27. $x-2+\frac{2}{x+2}$.
28. $x-3+\frac{1}{x+3}$.
29. $n+4-\frac{16}{n-4}$.
30. $n-5-\frac{10}{n+5}$.

EXERCISES

Write as fractions and simplify results:

1. $a + \frac{a}{x}$.

2. $b - \frac{b}{b+1}$.

3. $m + n + \frac{n}{m}$.

4. $a + x - \frac{a}{x}$.

5. $a - 3 + \frac{4}{a}$.

6. $2x + 3 - \frac{x^2}{2x}$.

7. $b - \frac{6}{b+3} - 3$.

8. $n - 2 - \frac{2n}{n-3}$.

9. $x - \frac{x+2}{x-2} + 1$.

10. $n - t + \frac{n^2 - t^2}{n+t}$.

11. $\frac{a+2}{2a+1} + 2a - 1$.

12. $\frac{x^2 + n^2}{3x - n} + 3x + n$.

13. $n - t + \frac{n^2 - 4t^2}{n+t}$.

14. $2n - x - \frac{n^2 + x^2}{2n + x}$.

15. $x^2 + x + 1 - \frac{x^3 + 2}{x - 1}$.

16. $x^2 + y^2 - \frac{x^3 + y^3}{x + y} - xy$.

17. $\frac{a}{a-2} - \frac{a-2}{a+2} - \frac{8}{a^2 - 4}$.

18. $\frac{b^2}{b^2 - 1} + \frac{b}{b+1} - \frac{b}{1-b}$.

19. $\frac{3a}{2+b} - \frac{a}{b-2} + \frac{1}{b^2 - 4}$.

20. $\frac{x}{x+1} + \frac{2x^2}{x^2 - 1} - \frac{2x}{x-1}$.

21. $x^2 + x - \frac{x^4 + 3x^2 + 1}{x^2 - x + 1} + 1$.

22. $\left(2a + 3 - \frac{4}{a-2}\right) - \left(a + 1 - \frac{3}{a+4}\right)$.

HINT. Removing parentheses,

$$2a + 3 - \frac{4}{a-2} - a - 1 + \frac{3}{a+4} = a + 2 - \frac{4}{a-2} + \frac{3}{a+4}, \text{ etc.}$$

23. $\left(5x + \frac{6x}{a}\right) - \left(x - \frac{x - 3a}{a}\right)$.

24. $\left(7x - \frac{2y}{3yz}\right) - \left(5x + \frac{x}{3yz}\right)$.

25. $\left(5n-\dfrac{2n}{x-y}\right)-\left(2n-\dfrac{4n}{x+y}\right).$

26. $\left(2n-3t+\dfrac{3n}{2n+t}\right)+\left(3n+2t-\dfrac{5n}{2n-t}\right).$

27. $\left(m+n-\dfrac{m}{m-n}\right)-\left(m-n+\dfrac{n}{m+n}\right).$

28. $\left(n-3-\dfrac{3n}{n+3}\right)-\left(n+3+\dfrac{3n}{n-3}\right).$

29. $\left(ax+a-\dfrac{a}{x+1}\right)-\left(ax-a-\dfrac{x}{x-1}\right).$

89. Multiplication of fractions. In algebra as in arithmetic the product of two or more fractions is the product of their numerators divided by the product of their denominators.

Thus, $\frac{3}{5}\cdot\frac{4}{11}=\frac{12}{55}.$

Similarly, $\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{bd},$

and $7\cdot\frac{4}{5}=\frac{7}{1}\cdot\frac{4}{5}=\frac{28}{5}.$

In like manner, $n\cdot\dfrac{a}{b}=\dfrac{n}{1}\cdot\dfrac{a}{b}=\dfrac{na}{b}.$

If a factor occurs one or more times in any numerator and in any denominator of the indicated product of two or more fractions, it should be canceled the same number of times from both, thus giving the product of the several fractions in lower terms.

Thus, $\dfrac{a}{\cancel{x}}\cdot\dfrac{\cancel{x}}{n}=\dfrac{a}{n}.$

EXAMPLE

Multiply $\frac{3x^2y^3}{4a^4}$ by $\frac{8a^3}{12xy}$.

Solution. $\frac{3x^2y^3}{4a^4}\cdot\frac{8a^3}{12xy}=\frac{\overset{xy^2}{\cancel{3}\,\cancel{x^2}\cancel{y^3}}}{\underset{a}{\cancel{4}\,\cancel{a^4}}}\cdot\frac{\overset{\cancel{2}}{\cancel{8}\,\cancel{a^3}}}{\underset{\underset{2}{\cancel{4}}}{\cancel{12}\,\cancel{xy}}}=\frac{xy^2}{2a}.$

To find the product of two or more fractions or mixed expressions we have the

RULE. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

ORAL EXERCISES

Find the product of

1. $\frac{1}{4}\cdot\frac{2}{3}$.

2. $6\cdot\frac{3}{4}$.

3. $\frac{5}{6}\cdot\frac{18}{25}$.

4. $(\frac{4}{9})^2$.

5. $\frac{1}{a}\cdot\frac{1}{x}$.

6. $\frac{a}{x}\cdot\frac{n}{c}$.

7. $\frac{a}{2x}\cdot\frac{x}{a}$.

8. $\frac{a^2}{n^2}\cdot\frac{2n}{a}$.

9. $\frac{a^3}{n^2}\cdot\frac{n}{a}$.

10. $\left(\frac{n}{a^2}\right)^2$.

11. $2\left(\frac{a^3}{2n}\right)^2$.

12. $2a\left(\frac{x}{a}\right)$.

13. $\frac{2n}{3}\cdot\frac{9}{4n}$.

14. $\frac{a^2-x^2}{a}\cdot\frac{a}{a+x}$.

15. $\frac{a^2-4}{3x}\cdot\frac{6x}{a-2}$.

16. $\frac{3x+n}{nx}\cdot\frac{n^2x^2}{3x-n}$.

17. $\left(\frac{a+x}{a}\right)^2\cdot a$.

18. $\left(\frac{n}{a+n}\right)^2(a+n)$.

19. $\frac{n^2-9}{3x^2}\cdot\frac{x}{n-3}$.

EXERCISES

Find the product of

1. $\frac{8}{9} \cdot \frac{12}{20}$.

2. $\frac{5}{21} \cdot \frac{14}{25}$.

3. $\frac{4}{5} \cdot \frac{3}{8} \cdot 10$.

4. $1\frac{3}{7} \cdot \frac{14}{5}$.

5. $2 \cdot \frac{1}{5} \cdot \frac{15}{22} \cdot 6$.

6. $1\frac{17}{18} \cdot 2\frac{4}{7}$.

7. $1\frac{3}{7} \cdot 4\frac{1}{5}$.

8. $2\frac{3}{5} \cdot \frac{15}{26}$.

9. $\frac{16}{35} \cdot 32 \cdot \frac{14}{24}$.

10. $\frac{10\,a^2n^2}{15\,n^3} \cdot \frac{12\,n}{7\,a^3}$.

11. $\frac{6\,an}{9\,x} \cdot \frac{12\,x^2y}{16\,n}$.

12. $\frac{25\,x^3}{6\,a^2} \cdot \frac{4\,a^2x^4}{5\,x^5}$.

13. $\left(\frac{2\,x}{3\,a}\right)^2 \cdot \frac{9\,a}{x}$.

14. $\left(\frac{2\,ax}{5\,n}\right)^2 \cdot \frac{10\,n}{4\,a^3x}$.

15. $\frac{5\,ax}{2\,n^2} \cdot \frac{6\,nx}{10\,a^2} \cdot \frac{n^2a}{x^3}$.

16. $\frac{a}{b} \cdot \frac{a^2x}{n^2} \cdot \frac{b^3n^3}{ax}$.

17. $\frac{16\,a^2}{9\,n^4} \cdot \frac{12\,n^6}{40\,a^2} \cdot \frac{15\,na^3}{4\,n^3}$.

18. $\left(\frac{2\,y}{3\,bz}\right)^2 \cdot \frac{12\,b^3}{16\,c^2yz^2} \cdot 8\,c^5z^3$.

19. $\frac{x}{4\,n^2}\left(\frac{n}{2\,x}\right)^2 \cdot 3\frac{1}{5}$.

20. $\left(\frac{1}{4}\right)^2 \cdot \left(\frac{2\,x}{n}\right)^3 \cdot \left(\frac{2\,n}{x}\right)^2$.

21. $\frac{x^4}{(2\,x)^3} \cdot \left(\frac{3\,a}{x}\right)^2 \cdot \left(\frac{2}{3}\right)^2$.

22. $\frac{6\,n}{x}\left(\frac{-2\,x}{3\,n}\right)^3\left(\frac{9\,n^2}{4\,x}\right)^2$.

23. $\frac{n+2}{n-2} \cdot \frac{n^2-4\,n+4}{n^2-4}$.

24. $\frac{a+x}{a-x} \cdot \frac{a^2-x^2}{a^2+2\,ax+x^2}$.

25. $\frac{n^2-4}{n+3} \cdot \frac{2\,n+6}{3\,n-6}$.

26. $\frac{x+7}{x^2-25} \cdot \frac{3\,x-15}{ax+7\,a}$.

27. $\frac{3\,x-12}{ax+3\,a} \cdot \frac{a^2x+3\,a^2}{nx-4\,n}$.

28. $\frac{5\,a+5\,c}{an-cn} \cdot \frac{an^2-cn^2}{a^2+ac}$.

29. $\frac{2\,n^2+6}{5\,x^4} \cdot \frac{10\,x^3}{3\,n^2+9}$.

30. $\frac{x^2-x-6}{x^2-4} \cdot \frac{x+2}{x-3}$.

31. $\frac{x^2-16}{2\,x^2-18} \cdot \frac{x^2+x-6}{x^2+x-20}$.

32. $\left(1+\frac{2-2a}{a^2-1}\right)\left(\frac{2a}{a-1}-1\right)$.

HINT. $\left(1+\frac{2-2a}{a^2-1}\right)\left(\frac{2a}{a-1}-1\right)=\frac{a^2-2a+1}{a^2-1}\cdot\frac{a+1}{a-1}$ etc.

33. $\left(1+\frac{3x}{1-2x}\right)\left(4-\frac{3}{1-x^2}\right)$.

34. $\left(\frac{6t-9}{2t-3}+2t\right)\left(2t-9+\frac{36}{2t+3}\right)$.

35. $\frac{9n^3}{3n^2-54n+51}\cdot\left(1+\frac{2}{n}-\frac{3}{n^2}\right)$.

36. $\frac{ax^2+2ax+4a}{6xy}\cdot\frac{12x^2y^2}{3x^2+6x+12}\cdot\left(3-\frac{4}{2-x}\right)$.

37. $\left(\frac{1}{a}+\frac{1}{a^2}-\frac{30}{a^3}\right)\left(a+\frac{20-9a}{5-a}-4\right)$.

38. $\left(1+\frac{1-7a}{a^2-1}\right)\left(\frac{5a^4+5a^3}{3a^2-33a+84}\right)\left(1-\frac{5}{a}+\frac{4}{a^2}\right)$.

39. $\left(n+3+\frac{n-3}{3n+5}\right)\left(4-3n-\frac{21}{n+4}\right)\left(\frac{1}{2n+2}\right)$.

90. Division of fractions. In arithmetic, $\frac{2}{5}\div\frac{4}{7}=\frac{2}{5}\cdot\frac{7}{4}=\frac{14}{20}=\frac{7}{10}$, and $\frac{3}{7}\div 8=\frac{3}{7}\cdot\frac{1}{8}=\frac{3}{56}$. Also, $\frac{5}{7}\div 2\frac{3}{4}=\frac{5}{7}\div\frac{11}{4}=\frac{5}{7}\cdot\frac{4}{11}=\frac{20}{77}$.

Similarly, $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$, and $\frac{a}{b}\div n=\frac{a}{b}\cdot\frac{1}{n}=\frac{a}{bn}$.

Also, $\frac{a}{b}\div\left(c+\frac{n}{d}\right)=\frac{a}{b}\div\left(\frac{cd+n}{d}\right)=\frac{a}{b}\cdot\frac{d}{cd+n}=\frac{ad}{bcd+bn}$.

For division of fractions we have the

RULE. *Reduce all integral or mixed expressions to fractional form.*

Then invert the divisor, or divisors, and proceed as in multiplication of fractions.

NOTE. *Inverting the divisor.* The student of algebra will remember that in arithmetic the divisor is inverted in division of fractions, but he may fail to recall just why. It is worth while for anyone to understand thoroughly this particular step of the process of division of fractions and why division is then replaced by multiplication. In a book on arithmetic the explanation could be made clear by analyzing the solution of a simple problem like $8 \div \frac{3}{5}$. To see just why the divisor is here inverted consider first $1 \div \frac{3}{5}$. This last requires that a unit of one kind (unity itself) be divided by three units of a different kind (fifths). Any difficulty of this nature is avoided by reducing the number 1 to fifths; then $1 \div \frac{3}{5}$ becomes $\frac{5}{5} \div \frac{3}{5}$, and the problem is transformed to that of dividing three units of a certain kind (fifths) into five of the same kind. This can be done directly and we have $\frac{5}{5} \div \frac{3}{5} = \frac{5}{3}$. But the answer $\frac{5}{3}$ is $\frac{3}{5}$ inverted. This means that to find how often $\frac{3}{5}$ is contained in the number 1 we merely invert it and obtain $\frac{5}{3}$. Obviously $\frac{3}{5}$ is contained in 2 twice as often as in 1, in 3 three times as often, and in 8 eight times as often, etc. Hence $8 \div \frac{3}{5} = 8 \times \frac{5}{3} = \frac{40}{3}$.

In the general problem $n \div \frac{a}{c}$ we may as before write $1 \div \frac{a}{c} = \frac{c}{c} \div \frac{a}{c} = \frac{c}{a}$. That is, $\frac{a}{c}$ is contained in the number 1, $\frac{c}{a}$ times. In n it will be contained n times as often as in 1. Hence $n \div \frac{a}{c} = n \times \frac{c}{a} = \frac{nc}{a}$.

EXERCISES

Perform the indicated operations:

1. $\frac{3}{2} \div \frac{4}{7}$.
2. $\frac{5}{51} \div \frac{10}{17}$.
3. $\frac{23}{39} \div \frac{115}{26}$.
4. $\left(4 + \frac{1}{5}\right) \div \left(4 + \frac{9}{10}\right)$.
5. $\left(4 - \frac{3}{4}\right) \div \left(4 + \frac{7}{8}\right)$.
6. $\left(7 + \frac{4}{11}\right) \div \left(1 - \frac{13}{22}\right)$.
7. $\frac{a}{b} \div \frac{c}{d}$.
8. $\frac{a}{b} \div \frac{a^2}{bc}$.

9. $\frac{3\,ab}{x} \div \frac{ac}{4\,nx}$.

10. $\frac{a^2}{n} \div \frac{2\,a}{n^2}$.

11. $\left(\frac{2\,an}{x}\right)^2 \div \frac{6\,an^2}{9\,x^4}$.

12. $\frac{a}{b} \div \frac{x}{n} \div \frac{an^2}{bx^2}$.

13. $\frac{3\,an}{4\,bx} \div \frac{6\,a^2n}{5\,b^2x} \div \frac{10\,an^2}{8\,cx}$.

14. $\frac{15}{4\,n^2} \div \frac{5\,a}{2\,n^6} \div \frac{(3\,n^2)^2}{12\,a^2}$.

15. $\frac{12\,y^3}{10\,x^2} \div \frac{6\,y}{5\,x} \div \frac{4\,ax}{3\,by^2}$.

16. $\frac{(4\,n^2)^2}{12} \cdot \left(\frac{2}{n}\right)^2 \div \frac{(2\,n)^2}{6}$.

17. $\frac{10\,ab^2}{21\,a^3} \div \frac{5\,b}{3\,a^2} \div 14\,a^4b$.

18. $\left(\frac{3\,a}{2\,x}\right)^2 \div \left(\frac{3\,a}{4\,x^2}\right)^2 \cdot \left(\frac{a}{x}\right)$.

HINT. See Rule, p. 207.

19. $\frac{(10\,n^2)^2}{256} \div \left(\frac{5\,n^2}{2}\right)^2 \div \frac{5\,n^3}{(8\,n^4)^2}$.

20. $\frac{a^2}{x} \div \left(\frac{x}{a}\right)^3 \div \left(\frac{a}{x^2}\right)^3 \cdot \left(\frac{1}{x^2}\right)^3$.

21. $\frac{2\,n-1}{15\,ax^3} \div \frac{6\,n-3}{5\,x}$.

22. $\frac{1-x^2}{x^2-16} \div \frac{x-1}{x^2-7\,x+12}$.

23. $\frac{n^3-n^2-12\,n}{n^2-4\,n+4} \div \frac{3\,n^2+n^3}{4-n^2}$.

24. $\left(3-\frac{1}{n+2}\right) \div \left(3-\frac{4}{n+3}\right)$.

25. $\left(x+\frac{n^2}{n+x}\right) \div \left(\frac{2\,nx+x^2}{n+x}-n\right)$.

26. $\left(\frac{n}{a}-\frac{9\,a}{n}\right) \div \left(\frac{n^2}{4\,a^2}-\frac{81\,a^2}{4\,n^2}\right)$.

27. $\left(x-3-\frac{28}{x}\right) \div \left(1-\frac{1}{x}-\frac{20}{x^2}\right)$.

28. $\left(n-\frac{3}{4\,n}+1\right) \div \frac{2\,n+3}{2\,n+1}\left(\frac{3}{4\,n^2-1}+3\right)$.

29. $\left(\frac{2\,x-6}{x+2}\right) \div \left(3+\frac{45}{4\,x^2-16}\right)\left(\frac{7}{x-3}+2\right)$.

30. $\left(\frac{a}{n}-\frac{n}{a}\right) \div \left(\frac{a+n}{2\,a^2-2\,an}\right) \cdot \frac{nx}{2\,(a-n)^2}$.

31. $\dfrac{9n^2+9n}{9n^2-4} \div \dfrac{n+1}{3n-2}\left(3n+4+\dfrac{4}{3n}\right).$

32. $\dfrac{4-4n^4}{4n^4-25n^2} \div \dfrac{2n^2+2}{2n-5}\left(3n^2+\dfrac{8n^2-n^3}{n-1}\right).$

33. $\left(3n^3-75n\right) \div \left(2+\dfrac{11}{n}+\dfrac{5}{n^2}\right)\left(6n-11-\dfrac{7}{n}\right).$

34. $\left(\dfrac{-6a}{a^2-4}+\dfrac{3}{2-a}\right) \div \left(\dfrac{3}{a^2-a-6}\right).$

35. $\left(4-\dfrac{6}{a+1}\right) \div \left(8-\dfrac{4a-8}{a^2-1}\right).$

36. $\left(\dfrac{2x}{x-2}-\dfrac{x}{x-1}\right) \div \left(\dfrac{3x}{x-3}-\dfrac{2x}{x-2}\right).$

37. $\left(\dfrac{8}{m-x}-\dfrac{4}{x^2-m^2}\right) \div \left(\dfrac{1}{m-x}-\dfrac{1}{m+x}\right).$

38. $\left(\dfrac{-5}{x^2-4}+\dfrac{2}{2-x}\right) \div \left(2+\dfrac{3}{x-2}\right).$

91. Complex fractions. A complex fraction is a fractional expression containing one or more fractions either in the numerator or in the denominator or in both.

EXAMPLE

Simplify $\dfrac{\dfrac{16}{x}-x}{\dfrac{24}{x^3}+\dfrac{10}{x^2}+\dfrac{1}{x}}.$

Solution. Reducing the numerator and denominator to simple fractions,

$$\frac{\dfrac{16}{x}-x}{\dfrac{24}{x^3}+\dfrac{10}{x^2}+\dfrac{1}{x}}=\frac{\dfrac{16-x^2}{x}}{\dfrac{24+10x+x^2}{x^3}}.$$

Performing the indicated division,

$$\frac{\dfrac{16-x^2}{x}}{\dfrac{24+10\,x+x^2}{x^3}} = \frac{(4-x)\cancel{(4+x)}}{\cancel{x}}\cdot\frac{\overset{x^2}{\cancel{x^3}}}{(x+6)\cancel{(x+4)}} = \frac{4\,x^2-x^3}{x+6}.$$

Check. If we let $x=2$,

$$\frac{\dfrac{16}{x}-x}{\dfrac{24}{x^3}+\dfrac{10}{x^2}+\dfrac{1}{x}} = \frac{4\,x^2-x^3}{x+6} \quad \text{becomes} \quad \frac{\dfrac{16}{2}-2}{\dfrac{24}{8}+\dfrac{10}{4}+\dfrac{1}{2}} = \frac{4\cdot 4-8}{2+6},$$

or

$$\frac{6}{6}=\frac{8}{8}.$$

For simplifying a complex fraction we have the

Rule. *Reduce the numerator and the denominator each to a simple fraction. Then multiply the numerator by the inverted denominator.*

EXERCISES

Simplify:

1. $\dfrac{4-\dfrac{1}{9}}{2-\dfrac{1}{3}}$.

2. $\dfrac{2+\dfrac{3}{5}}{4-\dfrac{1}{10}}$.

3. $\dfrac{\left(\dfrac{3}{5}\right)^2-1}{\dfrac{3}{5}+1}$.

4. $\dfrac{2^3-\left(\dfrac{1}{3}\right)^3}{3^2+1+\left(\dfrac{1}{3}\right)^2}$.

5. $\dfrac{\left(\dfrac{2}{5}\right)^3+\left(\dfrac{5}{3}\right)^3}{\left(\dfrac{2}{5}\right)^2-\left(\dfrac{2}{5}\right)\left(\dfrac{5}{3}\right)+\left(\dfrac{5}{3}\right)^2}$.

6. $\dfrac{\dfrac{4}{a}}{3-\dfrac{1}{a}}$.

7. $\dfrac{\dfrac{a}{b}}{\dfrac{b}{a}-\dfrac{a}{b}}$.

8. $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$.

9. $\dfrac{1-\dfrac{3}{a}-\dfrac{10}{a^2}}{a-13+\dfrac{40}{a}}$.

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10. $$\frac{\frac{1+x}{x}}{1-\frac{1}{x^2}}.$$

11. $$\frac{\frac{a}{b}}{c} \div \frac{a}{\frac{b}{c}}.$$

12. $$\frac{\frac{y}{x}-\frac{8\,x}{y}-2}{1+\frac{y}{x}-\frac{20\,x}{y}}.$$

13. $$\frac{1+\frac{x}{x-y}}{\frac{x}{x-y}}.$$

14. $$\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a}+\frac{1}{b}}.$$

15. $$\frac{x+\frac{1}{x+2}-2}{x+\frac{1}{x-2}+2}.$$

16. $$\frac{\frac{(x-2\,a)^2}{4\,ax}+2}{2+\frac{x}{a}}.$$

17. $$\frac{\frac{x}{1+x}-\frac{1-x}{x}}{\frac{x}{1+x}+\frac{1-x}{x}}.$$

18. $$\frac{\frac{a^2+b^2}{b}-2\,a}{1-\frac{b}{a}}.$$

19. $$\frac{\frac{1}{2\,x}+\frac{1}{y}}{\frac{1}{5\,y^2}+\frac{1}{2\,xy}+\frac{1}{5\,x^2}}.$$

20. $$\frac{\frac{1}{a^2}-\frac{1}{a^3}-\frac{12}{a^4}}{a-\frac{9}{a}}.$$

21. $$\frac{\frac{(x+5\,y)^2}{10\,xy}-2}{\frac{1}{y}-\frac{5}{x}}.$$

22. $$\frac{\frac{a}{3\,b}-\frac{3\,b}{a}}{\frac{4\,b}{a+b}-1}.$$

23. $$\frac{10+\frac{a}{x}}{\frac{(a-10\,x)^2}{20\,ax}+2}.$$

24. $$\frac{a+\frac{t^2}{a-2\,t}}{a-\frac{2\,t^2}{a+t}}.$$

25. $$\frac{8\,a-\frac{(3\,a+2\,x)^2}{3\,x}}{\left(\frac{3\,a+4\,x}{3\,a}\right)^2-\frac{6\,x}{a}}.$$

REVIEW EXERCISES

1. How are fractions added when they have the same denominators?

2. Is $\frac{5}{12}+\frac{7}{12}+\frac{1}{12}$ an operation of the type referred to in Exercise 1?

3. State an exercise of this type involving letters only.

4. In finding the sum of $\frac{3}{4}+\frac{4}{5}$, why is it not correct to add the denominators for the denominator of the result?

5. State the principle on which the changing of fractions to equivalent fractions in higher or lower terms depends.

6. What are equivalent fractions?

7. When we write $\frac{3}{5}+\frac{4}{7}=\frac{21}{35}+\frac{20}{35}$, on what fundamental principle is the equality based?

8. If 3 is added to the numerator and the denominator of $\frac{11}{15}$, will the value of the resulting fraction be equivalent to $\frac{11}{15}$? If 3 is subtracted?

9. If $\dfrac{a-3}{x-3}$ is transformed to $\dfrac{a}{x}$, what operations are really performed? Does this change the value of the fraction?

10. If $\dfrac{x+a}{3y+a}$ is changed to $\dfrac{x}{3y}$, what operations are really performed? Has the value of the fraction been changed?

11. Mention an error which must be avoided in simplifying the fraction

$$\frac{x+3+\dfrac{(x-2a)^2}{a}}{2x+4-\dfrac{x-2a}{a}}.$$

12. Give an example of a mixed number; of a mixed expression.

13. Why is it desirable to write

$$\frac{1}{(a-x)(x-b)}+\frac{3}{(x-a)(b-x)}$$

in the form $$\frac{-1}{(a-x)(b-x)}+\frac{-3}{(a-x)(b-x)}$$

before adding? By the use of what principle is the second form of the fractions obtained?

14. State the arithmetic method of multiplying $\frac{3}{5}$ by $\frac{4}{7}$. Is the process the same for algebraic fractions?

15. State the arithmetic method of dividing 5 by $\frac{3}{7}$. Is the process the same for the division of algebraic fractions?

16. What is a complex fraction?

17. How can the numerator of a complex fraction be distinguished from the denominator? Consider

$$\frac{\frac{a}{b}}{c}.$$

Which is the numerator? the denominator?

18. How is a complex fraction simplified?

19. In simplifying complex fractions like those on page 211, which of the four fundamental operations must be performed first? Which one is performed last?

20. In what two ways may $\frac{(\frac{2}{3})^2-1}{(\frac{2}{3})+1}$ be simplified?

21. Simplify $\frac{4-(\frac{3}{5})^2}{2+\frac{3}{5}}$ in the shortest way.

CHAPTER XVII

EQUATIONS CONTAINING FRACTIONS

92. Equations containing fractions with monomial denominators. If fractions are involved in one or both members of an equation, it is necessary to find a number or a literal expression by which one may multiply both members in order to get rid of the fractions. (Compare Example 1, p. 216.) This process involves the application of Axiom III, p. 57, which is the only principle employed in this chapter that has not been used repeatedly in the earlier work with equations.

Especial care is required to avoid errors when a fraction which has two or more terms in its numerator is preceded by a minus sign.

ORAL EXERCISES

Solve for x, stating what operations are necessary:

1. $\frac{x}{3} = 4.$
2. $\frac{x}{5} = 8.$
3. $\frac{x}{7} = 2.$
4. $\frac{x}{10} = 3.$
5. $\frac{2x}{3} = 4.$
6. $\frac{3x}{2} = 9.$
7. $\frac{3x}{5} = 6.$
8. $\frac{5x}{7} = 10.$
9. $\frac{x}{5} - 7 = 0.$
10. $\frac{x}{8} - 3 = 0.$
11. $\frac{2x + 4}{5} = 0.$
12. $\frac{3x - 6}{2} = 0.$

13. $\frac{3x-4}{2}=0.$

14. $\frac{x+1}{2}=3.$

15. $\frac{x-1}{3}=1.$

16. $\frac{x-2}{4}=5.$

17. $\frac{3x+2}{2}=4.$

18. $\frac{5x+2}{3}=4.$

19. $\frac{1}{x}=2.$

20. $\frac{2}{x}=1.$

21. $\frac{3}{x}=5.$

22. $\frac{5}{x}=8.$

23. $\frac{3}{2x}=5.$

24. $\frac{3}{5x}=15.$

25. $\frac{2}{x+1}=1.$

26. $\frac{5}{x-1}=1.$

27. $\frac{3}{x+1}=4.$

28. $4=\frac{1}{x-3}.$

29. $\frac{2}{x+1}=\frac{1}{x}.$

30. $\frac{a}{x}=\frac{x}{a}.$

31. Using the least multipliers possible, clear of fractions Exercises 1–10 on page 215.

EXAMPLES

1. Solve the equation $\frac{2x}{5}-\frac{x}{7}=9.$

Solution. Multiplying both members by 35 (L. C. M. of the denominators) and canceling, we obtain

$$14x-5x=315.$$

Then $$9x=315,$$

or $$x=35.$$

Check. Substituting 35 for x in the given equation,

$$\tfrac{70}{5}-\tfrac{35}{7}=9.$$

$$14-5=9.$$

2. Solve $\frac{x}{3} - \frac{3x}{5}\left(\frac{10}{x} - 4\right) + 3\frac{3}{5} = 14.$

Solution. Removing the parenthesis,

$$\frac{x}{3} - 6 + \frac{12x}{5} + \frac{18}{5} = 14.$$

Collecting,

$$\frac{x}{3} + \frac{12x}{5} = \frac{82}{5}.$$

Multiplying by 15 (the L. C. M. of the denominators) gives

$$5x + 36x = 246.$$

Then

$$41x = 246,$$

or

$$x = 6.$$

Check.

$$\frac{6}{3} - \frac{3 \cdot 6}{5}\left(\frac{10}{6} - 4\right) + 3\frac{3}{5} = 14.$$

$$2 - 6 + \tfrac{72}{5} + \tfrac{18}{5} = 14.$$

$$-4 + 18 = 14.$$

For solving equations containing fractions with monomial denominators, we have the

Rule. *Free the equation of any parentheses it may contain.*

Find the L. C. M. of the denominators of the fractions and multiply each term of the equation by it, using cancellation wherever possible.

Transpose and solve as usual.

EXERCISES

Solve and check:

1. $\frac{x}{2} + \frac{x}{5} = 14.$
2. $\frac{x}{3} + \frac{2x}{5} = 11.$
3. $\frac{2x}{5} + \frac{3x}{2} = 19.$
4. $3x - \frac{5x}{7} = 8.$
5. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 9.$
6. $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 3.$

7. $\frac{x+4}{2}-\frac{x-3}{5}=5.$

8. $\frac{3x+1}{4}+\frac{5x-1}{6}=8.$

9. $\frac{6}{t}-\frac{12+t}{3t}=\frac{4}{3}.$

10. $\frac{t-2}{3t}+\frac{5}{6}-\frac{3t-5}{4t}=0.$

11. $5t+\frac{1}{4}-4t\left(3-\frac{2}{t}\right)=3.$

12. $\frac{3+2t}{2t-3}=\frac{1}{4}.$

13. $\frac{t-5}{t+4}=\frac{7}{16}.$

14. $\frac{2t-3}{3t-2}=\frac{17}{28}.$

15. $\frac{2}{t-3}=\frac{1}{t-5}.$

16. $\frac{t-.6}{t+.4}=\frac{1}{6}.$

17. $\frac{.2t+4}{5}=\frac{.3t-3}{6}.$

18. $\frac{t+5}{t-9}=\frac{t+4}{t-3}.$

19. $\frac{5t-3}{4}+\frac{3}{4}(t-1)=\frac{5}{2}.$

20. $\frac{1}{2}\left(4-\frac{t}{2}\right)-\frac{3t-16}{6}=\frac{7-t}{6}.$

21. $\frac{t-3}{t-4}=\frac{t+5}{t+2}.$

22. $\frac{x+3}{x-5}=x-2.$

23. $\frac{x+5}{x-3}=x-4.$

24. $\frac{x}{x+1}+\frac{1}{x-1}=\frac{x+3}{x+1}.$

25. $8+\frac{x}{x-9}=x-\frac{7}{x-9}.$

26. $\frac{5x+1}{x-11}+8=2x.$

27. $\frac{t+4}{6t-2}=\frac{t}{2}+\frac{3t+10}{6}.$

28. $\frac{t+2}{t+3}+\frac{t+3}{t+2}=\frac{11t+28}{t^2+5t+6}.$

29. $\frac{x}{n}+n=1+x.$

30. $\frac{3x}{6}-\frac{x-3}{3}=\frac{19}{3}-\frac{x+4}{2}+10x.$

31. $\frac{4a}{5}+\frac{12(3a+1)}{2}=\left(\frac{a}{3}+2\right)+40+\frac{14}{15}.$

32. $\frac{x}{2}-2\left(\frac{4x}{5}-3\right)=4-\frac{3}{2}\left(\frac{x}{2}+1\right).$

33. $4-\frac{5x-15}{4}+\frac{2(x+2)}{3}=\frac{5(x-1)}{6}.$

34. $\frac{x}{n}+\frac{x}{a}=a+n.$

35. $\frac{n}{x}+\frac{3n}{2x}=\frac{5}{4}.$

36. $\frac{2cx}{n}-n^2+4c^2=\frac{nx}{2c}.$

37. $\frac{x}{a}+\frac{x}{n}+an=bn+ab+\frac{x}{b}.$

38. $\frac{x}{n}+3=\frac{x+2n}{a}+\frac{a}{n}.$

PROBLEMS

1. One sixth of a certain number, plus $\frac{1}{8}$ of the same number, equals 21. Find the number.

2. The difference between $\frac{1}{3}$ of a certain number and $\frac{1}{7}$ of the same number is 6. Find the number.

3. The sum of two numbers is 216. One tenth of the greater number equals $\frac{1}{8}$ of the less. Find the numbers.

4. The width of a rectangle is $\frac{3}{5}$ of its length. The perimeter is 200 feet. Find the area of the rectangle.

5. What number must be added to the numerator of the fraction $\frac{12}{25}$ so that the resulting fraction will be $\frac{1}{5}$ of the number added?

6. Three fourths of a certain integer is $\frac{1}{3}$ the sum of the next two consecutive integers. Find the first integer.

7. A certain even integer divided by 3 is 10 less than $\frac{1}{3}$ of the sum of the next two consecutive even integers. Find the first integer.

8. What number added to both terms of the fraction $\frac{5}{11}$ gives a fraction whose value is $\frac{3}{5}$?

9. Separate 96 into two parts such that $\frac{1}{6}$ of their difference is 5.

10. Separate 176 into two parts such that their quotient is $\frac{2}{9}$.

11. There are two numbers whose sum is 52. If their difference is divided by their sum, the quotient is $\frac{5}{13}$. Find the numbers.

12. The weight of Mars is approximately 9 times that of the moon, and the weight of the earth is about $\frac{81}{10}$ that of the moon and Mars combined. Find the weight of the moon and of Mars in terms of the earth's weight.

13. A's age is $\frac{7}{10}$ of B's age. In 4 years A's age will be $\frac{8}{11}$ of B's age. Find their ages now.

14. At the time of her marriage a certain woman's age was $\frac{2}{3}$ that of her husband. Twelve years later her age was $\frac{3}{4}$ of his. Find their ages at the time of their marriage.

15. A is 12 years older than B. Eight years ago B was $\frac{1}{2}$ as old as A. Find their ages now.

16. The denominator of a certain fraction exceeds its numerator by 12. If 6 is added to both terms of the fraction the value of the resulting fraction is $\frac{4}{5}$. Find the fraction.

17. Jupiter has 5 more moons than Uranus, and Saturn 2 more than twice as many as Uranus; Mars has 7 fewer than Jupiter, and Neptune $\frac{1}{2}$ as many as Mars. These planets together have 26 moons. How many has each?

18. A triangle has the same area as a trapezoid. The altitude of the triangle is 36 feet and its base is 20 feet. The altitude of the trapezoid is $\frac{1}{3}$ that of the triangle, and one of its bases equals the base of the triangle. Find the other base of the trapezoid.

19. A marksman hears the bullet strike the target 3 seconds after the report of his rifle. If the average velocity of the bullet is 1925 feet per second and the velocity of sound is 1100 feet per second, find his distance from the target and the length of time the bullet was in the air.

20. A gunner using a modern rifle would hear the projectile strike a target 2640 yards distant $9\frac{2}{5}$ seconds after the report of the gun, provided the projectile maintained throughout its flight the same velocity it had on leaving the gun. Find this velocity if sound travels 1100 feet per second.

21. The denominator of a certain fraction is 8 more than the numerator. If the numerator is increased by 1 and the denominator decreased by 3 the value of the resulting fraction is $\frac{2}{3}$. Find the fraction.

22. The estimated total possible water power of the world is 439 million horse power. Of this North America has 8 million horse power more than South America, 17 million more than Europe, 9 million less than Asia, and 45 million more than Oceanica. Africa has 59 million horse power less than all the others combined. Find the number of horse power for each geographical division.

93. Equations containing fractions with polynomial denominators. Although no new principle is involved in the exercises of this section, they serve to review some of the most important processes of algebra.

ORAL EXERCISES

Clear the following equations of fractions, stating in each case the operation employed. Do not solve the equations.

1. $\frac{x}{x-3}=2.$

Solution. Multiplying each member of the equation by $x-3$ gives $x=2x-6$.

2. $\frac{x}{x-5}=4.$
3. $\frac{2x}{4-x}=6.$
4. $\frac{3x}{2x+1}=5.$
5. $\frac{4x^2}{1-x^2}=3.$
6. $\frac{z-3}{z+3}=7.$
7. $\frac{z+8}{z-2}=8.$
8. $\frac{z-3}{z+5}=\frac{2}{3}.$
9. $\frac{z-6}{z+8}=\frac{3}{5}.$
10. $\frac{2y+1}{1-2y}=\frac{4}{7}.$
11. $\frac{y-3}{y+3}=\frac{3}{y}.$
12. $\frac{4}{x}=\frac{5-x}{5+x}.$
13. $\frac{3}{5x}=\frac{2x-1}{x+2}.$
14. $\frac{x-2}{x+1}=\frac{x-1}{x+2}.$
15. $\frac{x-3}{x-6}=\frac{x+6}{x+3}.$
16. $\frac{x-1}{2x}=\frac{x}{5}.$
17. $\frac{1}{t}=\frac{3}{2t-4}.$
18. $\frac{5}{x}=\frac{x}{7}.$
19. $\frac{3v-1}{2v+1}=\frac{2v-1}{1+3v}.$
20. $\frac{a}{x}+\frac{b}{x}=\frac{x}{a+b}.$
21. $\frac{x}{a}+\frac{a}{x}=2.$
22. $\frac{b}{y}+\frac{y}{b}=\frac{5}{y-b}.$

EXAMPLE

Solve the following equation:

$$3-\frac{2x-2}{3x+1}=2\tfrac{5}{11}.$$

Solution. Both members must be multiplied by $11(3x+1)$, the L. C. M. of the denominators.

This step may be indicated as follows:

$$11(3x+1)\left[3-\frac{2x-2}{3x+1}\right]=(2\tfrac{5}{11})11(3x+1).$$

Canceling where possible gives

$$33(3x+1)-11(2x-2)=27(3x+1);$$

whence $$99x+33-22x+22=81x+27.$$

Transposing and collecting, $-4x=-28,$

$$x=7.$$

EXERCISES

Solve and check as directed by the teacher:

1. $\dfrac{x+3}{x+1}=2.$

2. $\dfrac{x+2}{x+4}=3.$

3. $\dfrac{x-3}{2x-1}=\dfrac{2}{9}.$

4. $\dfrac{3x+4}{2x-1}=2.$

5. $\dfrac{4}{18-2x}=\dfrac{1}{x}.$

6. $\dfrac{3}{x-2}+\dfrac{5}{x}=2.$

7. $\dfrac{13}{x+5}+\dfrac{7}{2x}=2.$

8. $\dfrac{x-1}{x}=\dfrac{6x}{3x-2}.$

9. $\dfrac{x-2}{x+6}=\dfrac{x}{2}.$

10. $\dfrac{x}{5-2x}=\dfrac{2x}{8x-5}.$

11. $\dfrac{1}{3x+1}=\dfrac{2}{4+3x}.$

12. $\dfrac{1}{2-5x}=\dfrac{3}{3-5x}.$

13. $\dfrac{2x-5}{4-10x}+12x=0.$

14. $2x-\dfrac{2x+7}{3-2x}=3x.$

15. $\dfrac{x-3}{2x-5}=x-6\tfrac{5}{9}.$

16. $\dfrac{3y+4}{7-6y}=\dfrac{3y+2}{12y-1}.$

17. $\dfrac{14y-5}{4y+5}=\dfrac{12-2y}{2(3+2y)}.$

18. $\dfrac{15y-4}{8}+\dfrac{4}{5y-4}=\dfrac{15y}{8}.$

19. $\dfrac{3}{5-3y}=\dfrac{4-3y}{3y-5}+\dfrac{7}{5}.$

20. $\dfrac{7z}{z-2}=\dfrac{6z}{z-1}+6.$

21. $\dfrac{9y}{y-1} = \dfrac{16y}{2y-1} + 1.$

24. $\dfrac{z}{3z+2} - \dfrac{1}{7z-2} = 0.$

22. $\dfrac{3x^2+5x-2}{2x^2+5x+3} = \dfrac{3}{5}.$

25. $\dfrac{4}{x^2} = \dfrac{3}{x+1}.$

23. $\dfrac{3}{y} + \dfrac{y}{3} = 2\tfrac{1}{6}.$

26. $\dfrac{25}{x^2} = \dfrac{14}{3x-1}.$

27. $\dfrac{2}{2t+3} + \dfrac{1}{3-2t} = \dfrac{12t}{4t^2-9}.$

28. $\dfrac{2}{2t+1} + \dfrac{1}{2t-1} - \dfrac{8}{1-4t^2} = 0.$

29. $\dfrac{2}{x+8} = \dfrac{3}{5} - \dfrac{1}{2-x}.$

30. $\dfrac{1}{x+7} = \dfrac{2}{5} - \dfrac{1}{3-x}.$

31. $\dfrac{2t+1}{t^2+3t-4} = \dfrac{t+3}{t+4}.$

32. $\dfrac{t-1}{t^2-5t+6} = \dfrac{t-2}{2-t} - \dfrac{1+t}{t-3}.$

94. Equations involving decimals. The method of solving an equation containing decimals is illustrated in the following

EXAMPLE

Solve the equation $.9x + .7 = 4.2 - .15x.$

Solution. Multiplying each member of the equation by 100,

$$90x + 70 = 420 - 15x.$$

Transposing and combining,

$$105x = 350.$$

$$x = \tfrac{10}{3}.$$

Check.

$$.9 \times \tfrac{10}{3} + .7 = 4.2 - .15 \times \tfrac{10}{3}.$$
$$3 + .7 = 4.2 - .5.$$
$$3.7 = 3.7.$$

In equations containing fractions, if decimals occur in any denominator, it is often desirable to multiply both numerator and denominator of such a fraction by that power of 10 which will reduce the decimals in both the numerator and the denominator to integers. Then clear of fractions and proceed as in the foregoing example.

EXERCISES

Solve the following equations and check as directed by the teacher:

1. $.8\,x = 6.$

2. $.6\,x + .5 = .8.$

3. $.15\,x + 4 = .25.$

4. $.75 - .35\,x = .26.$

5. $.92 + .6\,x = 5.12.$

6. $3.75 = 2.15 - .25\,x.$

7. $.12\,x - 4.5 = 1.68.$

8. $.15\,x + .16 = .58.$

9. $.08\,x = .1\,x + 2.6.$

10. $.4\,x + 1.62 = .55\,x.$

11. $.3\,x - .8\,x = 16\,x - 49.5.$

12. $1.7\,x + 3.14 = -9.66 - 1.5\,x.$

13. $15\,x + 7 - 6.25\,x = 8.845 + 2\,x.$

14. $6\,x - 2.49\,x + 1.2\,x = 1.5 + .71\,x.$

15. $.36(2\,x + .5) - .6(1.5\,x - 2) = 1.2.$

16. $6(6\,x - 1.1) - 8.4(1.4\,x - 3) = 12\,x + .24.$

17. $.12(.5\,x + .05) - .15(.375\,x - 2) = .246.$

18. $\dfrac{.15\,x - 6.2}{4} - \dfrac{6.75 - .2\,x}{5} = \dfrac{3.5}{2}.$

19. $$\frac{6.2\,x - 1.24}{.4} - \frac{7.56 - .28\,x}{.35} = 7.9.$$

HINT. Multiplying numerator and denominator of each fraction on the left by 100,

$$\frac{620\,x - 124}{40} - \frac{756 - 28\,x}{35} = 7.9.$$

To avoid the possibility of repeating, in the check, a numerical error made in the solution, the check should be performed by finding the value of each fraction separately, without clearing of fractions.

20. $$\frac{.9\,x - .1}{1.7} = \frac{2.5 - .35\,x}{1.8}.$$

21. $$\frac{6.4\,x}{.5} + \frac{9\,x}{12.5} = 1.52 + 12\,x.$$

22. $$\frac{.15(3 - .4\,x)}{.16} - \frac{.25(.2\,x - 6)}{.8} = 2.5.$$

23. $$\frac{.3(10 - x)}{13.5} = \frac{1.5 - 5\,x}{48.5} + .28\,x - 3.8.$$

24. $$\frac{33}{3(x + 5)} + \frac{3.75}{1.5(x - 8.5)} = 0.$$

25. The formula $\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{x}$ applies to a convex lens where f_1 is the distance of an object from the lens, f_2 is the distance of the image formed by the lens, and x is a distance known as the focal length determined by the shape of the lens. If $f_1 = 3.4$ inches and $f_2 = 4.2$ inches, find x to two decimal places.

HINT. Solve the equation for x before substituting the given values.

26. If a pounds of water at a temperature t is mixed with b pounds of water at a temperature T, the temperature x of the resulting mixture (if no heat escapes) is given by the equation $at + bT = x(a + b)$. Find x if $a = 4\frac{1}{8}$ pounds, $t = 40°$, $b = 3\frac{1}{2}$ pounds, and $T = 90°$.

27. The velocity of sound is v in the equation $v = nl$ when n is the number of vibrations the sound makes per second and l is the length of one sound wave. The A to which an orchestra tunes makes 440 vibrations per second. If sound has a velocity of 1100 feet per second, find the wave length of the note A.

28. If a body were projected away from the earth's surface with a velocity in feet per second greater than $v = \sqrt{2gR}$ it would never return to the earth. If $g = 32$ and R equals the radius of the earth (4000 miles) expressed in feet, find this velocity.

NOTE. The introduction into Europe of the Arabic notation for numbers was one of the important events of the Middle Ages. This notation originated among the Hindus probably as early as 300 B.C. It was adopted by the Arabs, and was introduced by the Moors into Spain during the eighth century. Anyone who has tried to multiply two numbers in the Roman notation, like MDCCVII by MCXVIII, will realize the difficulties that surrounded arithmetical operations before the Arabic system was taught. Before the introduction of this system one of the principal uses of arithmetic was the determination of the day of the month on which Easter came. Roger Bacon in the thirteenth century urged the theologians "to abound in the power of numbering," so that they might carry out these computations. Business computations were made on the abacus, one form of which is a contrivance of wires and sliding balls on which arithmetical operations can be performed with great rapidity.

Though computation in the decimal system was used in Europe from the thirteenth century, the final step in perfecting the notation was not taken until about 1600, when Sir John Napier and others made use of the decimal point in the modern sense. It was not until the beginning of the eighteenth century that it came into general use.

95. Formula for percentage. The methods of algebra may be used to advantage in dealing with many problems in percentage which are also found in arithmetic, and in solving many others which would be difficult or impossible to solve by arithmetical means alone.

ORAL EXERCISES

1. What is 5% of 60?
2. What is 5% of x?
3. What is 4% of $x + 40$?
4. What is 7% of $4x - a$?
5. What is the interest on $100 invested for 1 year at 6%?
6. What is the interest on $100 invested for 6 years at 6%?
7. What is the interest on $100 invested for t years at 4%?
8. What is the interest on P dollars invested for t years at 4%?
9. What is the interest on P dollars invested for t years at r%?
10. What is the total amount due at the end of n years if P dollars are invested at r%?

If two sums of money are x dollars and $(800 - x)$ dollars respectively, express as equations the statements made in Exercises 11–14.

11. Four per cent of the first sum equals $20.
12. Five per cent of the first sum plus 4% of the second sum equals $120.

13. Six per cent of the first sum equals 5% of the second.

14. Five per cent of the first sum is $24 more than 4% of the second.

When P dollars are invested at simple interest for t years, at a yearly rate of r%, the total amount, A, accumulated is given by the following formula:

$$P + P \cdot r \cdot t = P(1 + rt) = A.$$

It should be noted carefully that the value of r is a fraction of which the denominator is 100. Thus, if the rate is 6%, the value of r is $\frac{6}{100}$, or .06.

EXAMPLE

What sum of money placed at simple interest for 3 years at 6% will amount to $236?

Solution. Principal + interest = $236.

Let $P =$ the principal in dollars.

Then $.06\,P \times 3 =$ the interest for three years in dollars.

Therefore $P + .18\,P = 236,$

or $1.18\,P = 236;$

whence $P = 200.$

Check. $200 + 200 \times .06 \times 3 = 236.$

EXERCISES

1. What sum of money placed at interest for 1 year at 6% will amount to $318?

2. What sum of money placed at simple interest for 3 years at 5% will amount to $368?

3. In how many years will \$475, at 4 % simple interest, gain \$76?

4. In how many years will \$650, at $4\frac{1}{2}$ % simple interest, gain \$29.25?

5. At what per cent simple interest will \$420 gain \$126 in 5 years?

Solution. \$420 × the rate of interest × 5 = \$126.

Let $r =$ the rate of interest.

Then $\$420\,r =$ the interest for one year,

and $\$420 \cdot r \cdot 5 =$ the interest for 5 years.

Therefore $\$2100\,r = \$126,$

and $r = \frac{6}{100}.$

Hence the interest rate is 6 %.

6. At what per cent simple interest will \$960 gain \$192 in 4 years?

7. At what per cent simple interest will \$250 amount to \$332.50 in 6 years?

8. In how many years will \$200 double itself at 5 % simple interest? \$300? x dollars?

9. In how many years will \$400 treble itself at 6 % simple interest?

10. A part of \$1000 is invested at 5 % and the remainder at 6 %. The yearly income from the two investments is \$53. Find each investment.

HINT. One part × .06 + the other part × .05 = \$53.

Let $x =$ the number of dollars invested at 6 %.

Then $1000 - x =$ the number of dollars invested at 5 %.

Therefore, by the conditions of the problem,

$$.06\,x + .05(1000 - x) = 53.$$

11. A part of \$1500 is invested at 5% and the remainder at 4%. The total annual income from the two investments is \$66. Find the amount of each investment.

12. A sum of money at 6% interest and a second sum at 5% yield a total annual income of \$61. The first sum exceeds the second by \$100. Find each.

13. A 4% investment yields annually just as much as one at 5%. If the sum of the investments is \$2700, find each.

14. A 5% investment yields annually \$7 more than a 6% investment. If the sum of the two investments is \$800, find each.

15. A man invests part of \$4300 at 6% and the remainder at 5%. The investment at 6% yields annually \$32.50 more than the one at 5%. Find the sum invested at 5%.

16. A man invests part of \$5450 at 5% and the remainder at 6%. The yearly income from the 5% investment is \$3 more than that from the 6% investment. Find the sum invested at 6%.

17. A part of \$5000 is invested at 4% and the remainder at 5%. The total yearly income is \$220. Find the amount invested at 5%.

18. An atom of hydrogen is .063 as heavy as an atom of oxygen. Each molecule of water is made up of two atoms of hydrogen and one of oxygen. What percentage of the weight of a molecule of water is hydrogen? How many pounds of oxygen are in 100 pounds of water?

96. Literal equations. At this point the student should review the work on pages 124–126.

EXERCISES

Solve for x, y, z, or t and check as directed by the teacher:

1. $3\,ax - 4\,a^2 = 28\,a^2 - 5\,ax.$

2. $39\,c^2 + 5\,cx = 2\,c(6\,x - 5\,c).$

3. $5\,a^2x - 3\,b^2 = 2\,b^2 - 3\,a^2x - 5\,b^2.$

4. $3(x + 5) - 6\,a = 9.$

5. $5(x - 2) - 10\,c = 20.$

6. $3(x - 2) + 4(2 - 3\,x) = 20 - 18\,a.$

7. $5(2\,x - a) - 4(x - a) = \frac{18}{a} - a.$

8. $\frac{y}{bc} + \frac{y}{ac} = \frac{a}{bc} + \frac{1}{c}.$

9. $3\,b(8 - 4\,a) = 4\,a(12\,y - 3\,b).$

10. $\frac{y}{ab} + \frac{b}{ac} = \frac{y}{ac} + \frac{1}{a}.$

11. $a - \frac{a^2 + 3\,ab}{z + a} = \frac{z^2 + a^2}{z + a} - z.$

12. $\frac{a^2z + 3\,az}{a^2 - a + 1} - 1 = \frac{z - 10\,a + 2}{a^2 - a + 1} + z.$

13. $\frac{a}{t} + \frac{3\,a}{2\,t} = \frac{5}{2}.$

14. $\frac{8\,a}{t} + \frac{24\,a}{t} = \frac{3}{2} + \frac{5\,a}{t}.$

15. $\frac{a^2}{x} - a = \frac{b^2}{x} + b.$

16. $\frac{a}{x} + \frac{x}{a} = 2.$

17. $\frac{6\,x}{a} - \frac{6\,a}{x} = 5.$

18. $10\,x - \frac{5\,b}{a} = \frac{2\,a}{b} - 1.$

19. $\frac{x + c}{3} = \frac{x + 3}{c}.$

20. $\frac{x - m}{a} = \frac{x - a}{m}.$

21. $\dfrac{z-a^2}{z-c^2}=\dfrac{a}{c}.$

22. $\dfrac{abc+c}{ab}+\dfrac{c}{abz}=\dfrac{abc}{z}.$

23. $\dfrac{a^2}{cz}-\dfrac{c^2}{az}=\dfrac{1}{a}-\dfrac{1}{c}.$

24. $\dfrac{a+t}{2(c+t)}-\dfrac{a-t}{2c-2t}=\dfrac{1}{t^2-c^2}.$

25. $\dfrac{a-c}{3ac}-\dfrac{a-c}{z}=\dfrac{c^2}{3az}-\dfrac{a^2}{3cz}.$

26. $x-\dfrac{ax}{x+a}=\dfrac{1}{x+a}.$

27. $\dfrac{t-a}{a}-\dfrac{a}{t-a}=\dfrac{3}{2}.$

28. $\dfrac{a}{2a-t}+\dfrac{2a-t}{a}=2.$

29. $\dfrac{t+9a}{a}=\dfrac{t}{m-a}-\dfrac{m-4a}{a-m}.$

30. $\dfrac{t^2+1}{t}=a+\dfrac{1}{a}.$

31. $\dfrac{m^2}{x^2}=\dfrac{m+1}{x+1}.$

32. $\dfrac{2r-1}{2x-1}=\dfrac{r^2}{x^2}.$

97. Meaning of primes and subscripts. Different but related values are often represented by the same letter with smaller characters written at the right and above or below the letter used, as y', y'', x_0, $4x_3$, t_m^2, t_w. These are read *y prime, y second, x sub zero,* 4 *x sub three, the square of t sub m,* and *t sub w* respectively. Primes and subscripts, unlike exponents, possess no numerical significance, and the student should carefully note that x_0 and x_3 are as different numerically as a and b.

The notation just explained is very convenient in physics, where L_1 and L_2 may denote different but related lengths; W_1 and W_2 may represent two different weights; and t_0, t_1, and t_2 may mean three unequal but related intervals of time.

Primes are cumbersome and easily confused with exponents; hence subscripts are preferable.

The following equations are taken from algebra, geometry, and physics, where it is often necessary to express some one quantity (weight, time, distance, etc.) in terms of others.

EXERCISES

1. Solve for R; $K = 2\,\pi RH$.
(Formula for curved surface of cylinder.)

2. Solve for a; $A = \frac{ab}{2}$.
(Formula for area of triangle.)

3. Solve for R; $C = 2\,\pi R$.
(Formula for circumference of circle. $\pi =$ approximately $\frac{22}{7}$.)

4. Solve for r and for t; $d = rt$.
(Formula for uniform motion.)

5. Solve for a and for A; $\frac{a}{A} = \frac{D}{360}$.
(Formula relating to the measurement of angles.)

6. Solve for C; $\frac{D}{360} = \frac{1}{C}$.

7. Solve for r; $I = \frac{E}{R + r}$.
(Ohm's law for a simple electrical circuit.)

8. Solve for r and for n; $I = \frac{E}{R = nr}$.

9. Solve for r and for n; $I = \dfrac{n \cdot e}{R + nr}$.

10. Find I in Exercise 9, if $n = 4$, $e = 1$, $R = 5$, and $r = 1.2$.

11. Solve for F; $C = \frac{5}{9}(F - 32)$.

(Formula for converting thermometer readings from one scale (Fahrenheit) to another (centigrade).)

12. Solve for W_2; $\dfrac{W_1}{W_2} = \dfrac{L_2}{L_1}$.

13. Solve for r and for t; $A = P(1 + rt)$.

14. Solve for P_2; $\dfrac{V_1}{V_2} = \dfrac{P_2}{P_1}$.

(Formula relating to volume and pressure of a gas.)

15. Solve for n and for l; $s = \dfrac{n(a + l)}{2}$.

16. Find n in Exercise 15, if $a = 1$, $l = 100$, and $s = 5050$.

17. Solve for a, l, and r; $s = \dfrac{rl - a}{r - 1}$.

18. Solve for a; $\dfrac{D}{180} = \dfrac{a}{n}$.

19. Solve for t_1; $V_1 = V_0(1 + .00365\, t_1)$.

20. Solve for b_2; $A = \dfrac{(b_1 + b_2)a}{2}$.

21. Solve for a, b, and f; $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{f}$.

22. Find f in the formula of Exercise 21 if $a = 10$ and $b = 18.5$.

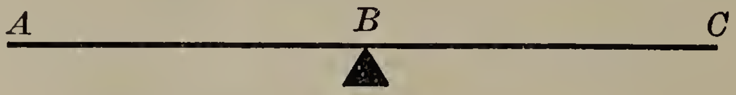
23. The formula for the area of a circle is $A = \pi r^2$. $\pi = 3.1416$. If $r = 10$ inches, find A.

24. The formula for the volume of a cone is $V = \dfrac{\pi r^2 h}{3}$. Find V for a cone in which $r = 6$ inches and $h = 15$ inches.

25. Solve for W; $E = \frac{WV^2}{2g}$.

26. In Exercise 25, if g is 32.2, $V = 40$, and $E = 10,000$, find W.

98. The lever. The figure given below is a diagram of a machine called a *lever*. AC is a stiff bar resting on a single support at B. This support is called the *fulcrum* and AB and BC are spoken of as *arms* of the lever.



Those who have played with a teeter board have had some experience with a lever, and they have found that, in order to balance, the heavier of two persons must sit nearer the fulcrum than the lighter one does.

In general, if the lengths of the arms of a lever are l_1 and l_2 and the corresponding weights are W_1 and W_2, a balance results when

$$l_1 W_1 = l_2 W_2.$$

Thus, if $AB = 3$ feet and $BC = 4$ feet, a boy at A who weighs 100 pounds will balance a boy at C who weighs 75 pounds; for $3 \cdot 100 = 4 \cdot 75$.

PROBLEMS

1. A, who is 5 feet from the fulcrum, balances B, who is 7 feet from it. A weighs 102 pounds. Find the weight of B.

2. A, who weighs 100 pounds, balances B, who weighs 120 pounds. B is 6 feet 3 inches from the fulcrum. How far from it is A?

3. A, who weighs 120 pounds, balances B, who weighs 100 pounds. The distance between them is 11 feet. How far is each from the fulcrum?

4. A and B together weigh 315 pounds. They balance when A is 3 feet 9 inches from the fulcrum and B is 5 feet from it. Find the weight of each.

5. A weighs $\frac{2}{3}$ as much as B. The distance between them is 12 feet. How far from B is the fulcrum if they balance?

In each of the preceding problems the fulcrum may be thought of as the *center of gravity* of the two objects balanced.

6. The center of the moon is approximately 238,000 miles from the center of the earth. The earth is 81.5 times as heavy as the moon. Find the distance from the earth's center to the center of gravity of the earth and moon.

99. Ratio. The *ratio* of one number to a second number is the quotient obtained by dividing the first number by the second.

Thus the ratio of 1 to 2 is $\frac{1}{2}$; that of 7 to 3 is $\frac{7}{3}$; that of a to b is $\frac{a}{b}$.

Hence every ratio is a fraction and all fractions may be regarded as ratios. The ratio $\frac{a}{b}$ is often written $a : b$. The symbol (:) may be thought of as the sign for division (÷) abbreviated by omitting the bar.

EXERCISES

Simplify the following ratios by writing them as fractions and reducing the fractions to their lowest terms:

1. $2 : 4$.
2. $3 : 12$.
3. $4 : 12$.
4. $\frac{1}{8} : \frac{3}{24}$.
5. $13 : 15$.
6. $14\,a : 21\,a^2$.
7. $13\frac{1}{2} : 54$.
8. $3.6 : 12$.
9. 12 feet : 12 inches.
10. 1 mile : 264 feet.
11. $3\frac{1}{2}$ pounds : 12 ounces.
12. $2.45 : 1.75$.

13. Separate 35 into two parts which are in the ratio $2:3$.

HINT. Let one part be represented by $2x$; then the other will be $3x$.

14. Separate 24 into two parts which are in the ratio $5:7$.

15. Separate 121 into two parts which are in the ratio of 10 to 1.

16. Separate 312 into three parts which are in the ratio of 2 to 4 to 7.

17. What number added to both terms of the ratio of 12 to 5 gives a result equal to the ratio of 2 to 1?

18. What number subtracted from both terms of the ratio of 13 to 27 gives a result equal to the ratio of 4 to 11?

19. If y is a positive number, which is the greater ratio,

$$\frac{2}{5} \text{ or } \frac{2+y}{5+y}? \qquad \frac{y}{y+3} \text{ or } \frac{2y}{2y+3}?$$

HINT. Reduce the two fractions in each part to respectively equivalent fractions having a common denominator, and then compare the numerators.

20. From your answers to Exercise 19 state the change which occurs in the value of a proper fraction when a positive number is added to both its terms.

21. If x is a positive number, which is the greater ratio,

$$\frac{10}{7} \text{ or } \frac{10+x}{7+x}? \quad \frac{1+x}{x} \text{ or } \frac{1+2x}{2x}?$$

22. From your answers to Exercise 21 state the change which occurs in the value of an improper fraction when a positive number is added to both its terms.

100. Proportion. A *proportion* is a statement of equality between two ratios.

Thus $\frac{2}{4} = \frac{3}{6}$, $\frac{4}{5} = \frac{12}{15}$, and $\frac{10}{20} = \frac{1}{2}$ are proportions, for the ratios are equal in each case.

The four numbers 1, 2, 3, and 6 are said to be in proportion, for the ratio of the first pair equals the ratio of the second pair. In general, the numbers a, b, c, and d are in proportion if

$$a : b = c : d. \qquad (1)$$

In (1), a and d (the first and fourth terms) are called *extremes*, and b and c are called *means*.

Since a proportion is an equality between two ratios (fractions), it is therefore an equation. Hence *any operation which may be performed on an equation may be performed on a proportion.* (See Axioms, pp. 56–58.)

Thus, in the proportion $\frac{a}{b} = \frac{c}{d}$ both members may be multiplied by bd, giving $ad = bc$. Here the first member is the product of the extremes of the proportion, and the second member is the product of the means.

Therefore, *in any proportion the product of the extremes equals the product of the means.*

EXERCISES

Form proportions from the following by supplying the missing terms:

1. $\frac{2}{3} = \frac{12}{?}$.
2. $\frac{1}{2} = \frac{4}{?}$.
3. $\frac{12}{10} = \frac{?}{5}$.
4. $\frac{36}{40} = \frac{9}{?}$.
5. $\frac{15}{?} = \frac{3}{4}$.
6. $\frac{13}{52} = \frac{1}{?}$.
7. $\frac{5}{7} = \frac{?}{35}$.
8. $\frac{2}{9} = \frac{?}{81}$.
9. $\frac{12}{3} = \frac{4}{?}$.

Solve the following for x:

10. $12 : 7 = x^2 : 21.$

11. $\frac{3x+1}{2x+1} = \frac{7}{5}.$

12. $3 : \frac{x}{2} = \frac{x}{2} : 12.$

13. $5 : 1 = 1 : \frac{1}{x}.$

14. $m : 1 = x : n.$

15. $\frac{1}{x} = \frac{x}{25}.$

16. $\frac{4}{3} = \frac{12}{x^2}.$

17. $x : a = b : c.$

18. $\frac{6}{x} = \frac{5x+9}{12}.$

19. $r : x = r : s.$

20. $\frac{2x}{3} : p = \frac{p}{2} : q.$

21. $x : 1 = 1 : v.$

101. Variation. If the price of a certain kind of cloth is 50 cents a yard, the number of cents, A, that n yards would cost is given by the formula $A = 50n$. In this case the ratio $A : n = 50$. If n_1 and n_2 denote the number of yards of cloth bought for A_1 and A_2 cents respectively, then $\frac{A_1}{n_1} = \frac{A_2}{n_2} = 50$. In other words, the amount of money paid for the cloth is proportional to the number of yards of the cloth purchased.

Another way of expressing this idea is to state that the total cost of the cloth *varies directly* as the number of yards of cloth purchased.

When one variable varies directly as another, the ratio of any corresponding pair of values of the variables is constant.

The volume of gas in a tank varies as the temperature. That is, if V_1 and V_2 are the volumes corresponding to the temperatures t_1 and t_2, then $\frac{V_1}{t_1} = \frac{V_2}{t_2}$. Furthermore, if V is the volume at *any* temperature t, and V_1 is the volume at a particular temperature t_1, then $\frac{V}{t} = \frac{V_1}{t_1}$, or $V = \left(\frac{V_1}{t_1}\right)t.$

Since V_1 and t_1 are definite numbers, $\frac{V_1}{t_1}$ is a constant. This illustrates the fact that *if V varies as t, then* $V = kt$.

That is, the statement that any number, as x, varies as another number, as y, is equivalent to the equation $x = ky$, where k is a constant. If the value of this constant, and a particular value of either x or y, are known, then the value of the other variable can be found.

In the first illustration above, $A = 50\,n$. That is, $k = 50$. If A, the amount of money expended, is known, then the number of yards of cloth can be found.

EXAMPLES

1. A train travels at an average rate of 40 miles per hour. Express this as a formula, and tell what variables vary directly as each other.

Solution. Let d denote the number of miles traveled by the train in t hours.

Then $$d = 40\,t, \text{ or } \frac{d}{t} = 40.$$

Here the number of miles traveled varies directly as the time.

2. The ratio of the area of a circle to the square of its radius is constant and equal to the number π $(= 3.14 +)$. Express this as a formula and tell what variables vary directly as each other.

Solution. Let A denote the area of any circle and r its radius.

Then $$A = \pi r^2, \text{ or } \frac{A}{r^2} = \pi.$$

The area of a circle varies directly as the square of its radius.

EXERCISES

In the following exercises determine which are true proportions:

1. $3 : 7 = 6 : 15$.

HINT. Express the ratios in fractional form.

2. $3 : 5 = 24 : 40$.

3. $25 : 3 = 75 : 10$.

4. $2 : 3 = 14 : 20$.

5. $3 : 4 = 36 : 48$.

6. $15 : 9 = 5 : 3$.

7. Two workmen are paid "piecework rates"; that is, they receive a certain sum of money for each article they make. If the rate is 2 cents per piece, state a formula showing what their earnings will be for n articles. If one man produces 150 pieces and the other produces 130 pieces an hour, what will their hourly earnings be?

8. A certain map is drawn so that it represents actual distances in the proportion of 1 inch to 1 mile. How long would a lake 5 miles long appear on the map? A river 500 feet wide would have what width on the map?

9. The cost of linoleum is proportional to the area bought. If a piece 4 feet by 12 feet cost $12, what was the rate per square foot? How much would 100 square feet cost?

10. A man invests $5000 and receives $200 annual interest. How much interest will he receive from $4500? How much must he invest to receive $350 per year at the same interest rate? What was the rate of interest?

11. The distance (S), in miles, traveled by a train moving uniformly is expressed by the formula $S = VT$, in which V is the rate of the train in miles per hour, and T is the time in hours during which the train is moving. If the train travels 100 miles in $2\frac{1}{2}$ hours, how far can it go in 4 hours? How long will it take it to cover 125 miles? What is its speed?

12. The cost of a building is often expressed in terms of its cost per cubic foot of capacity. If the rate is 50 cents per cubic foot, how much will a building 100 feet long, 50 feet wide, and 40 feet high cost? A building 75 feet by 75 feet by 100 feet cost $562,500 to construct. What was the rate per cubic foot?

Write each of the following rules as a direct variation, using letters to represent the quantities involved:

13. Interest varies directly as time.

14. The area of a rectangle varies directly as the product of the length times the width.

15. The interest on a sum of money varies directly as the interest rate.

16. The speed with which a body is falling at any instant varies directly as the length of time during which it has been falling.

17. The usual cost of a railway ticket varies directly as the distance to be traveled.

18. The weight of any given size of wire is proportional to the length of the wire. A certain size of wire has 100 feet to the pound. How much wire is there in a roll weighing 75 pounds?

19. The volume of a sphere is proportional to the cube of the radius. What is the ratio between the volumes of two spheres one of which has a diameter $\frac{1}{2}$ as great as the other?

20. The area of a circle is proportional to the square of the radius. What is the area of a circle of 2-foot radius if a circle of 1-foot radius has an area of 3.14 square feet?

21. A man pays 4 cents a mile for a railway ticket, and the total cost is \$8. How far is he traveling? Express the relation by means of a formula.

22. The area of the surface of a cube is proportional to the square of any edge. What is the edge of a cube 6 square feet in area if a cube with an edge of 3 feet has an area of 54 square feet?

REVIEW PROBLEMS

1. Separate 318 into two parts such that their quotient is 7.

2. Separate 170 into two parts such that $\frac{2}{3}$ of the greater shall equal $\frac{3}{4}$ of the less.

3. Separate $\frac{5}{3}$ into two parts such that $\frac{1}{3}$ of one part shall equal $\frac{1}{5}$ of the other.

4. Find two numbers whose sum is 162 and such that the greater divided by the less gives a partial quotient of 3 and a remainder of 14.

HINT. $\dfrac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \dfrac{\text{Remainder}}{\text{Divisor}}.$

Let $x =$ the less number.

Then $162 - x =$ the greater number.

That is, $\dfrac{162 - x}{x} = 3 + \dfrac{14}{x}.$

Solve and check as usual.

5. Separate 149 into two parts such that one divided by the other gives a partial quotient of 4 and a remainder of 4.

6. The sum of two numbers is 1516. The greater divided by the less gives a partial quotient of 5 and a remainder of 130. Find the numbers.

7. Separate $\frac{11}{5}$ into two parts such that their product is greater by $\frac{12}{25}$ than the square of the smaller part.

8. The sum of two numbers is 36. Four times the greater number exceeds 50 by twice as much as 36 exceeds the less. Find the numbers.

9. A boy's age now is $\frac{3}{5}$ of what it will be 8 years hence. How old is he now?

10. One sixth of a certain man's age 8 years ago equals $\frac{1}{10}$ of his age 8 years hence. What is his age now?

11. A collection of nickels and quarters contains 60 coins. Their total value is $11.60. How many are there of each?

12. Twenty-three coins, dimes and quarters, have the value $4.70. How many are there of each?

13. The square of half a certain even number is 92 less than $\frac{1}{4}$ the product of the next two consecutive even numbers. Find the numbers.

14. A rectangle is four times as long as it is wide. If it were 8 feet shorter and 3 feet wider, its area would be 104 square feet more. Find its length and breadth.

15. A rectangle is $\frac{2}{3}$ as broad as it is long. If its length were doubled and its breadth diminished by 32, its area would be 1584 square feet. What are its dimensions?

16. It costs as much to sod a square piece of ground at 20 cents per square yard as to fence it at 15 cents per running foot. Find the side of the square.

17. A rectangular court is twice as long as it is wide. It costs half as much to fence it at $66\frac{2}{3}$ cents per running foot as to seed it at 5 cents per square yard. Find its dimensions.

18. A rectangular picture twice as long as it is wide is surrounded by a frame 3 inches wide. The area of the frame is 288 square inches. Find the dimensions of the picture.

19. A man bought apples at 16 cents per dozen. He sold $\frac{1}{4}$ of them at the rate of 3 for 7 cents, but the rest were not so good, and he had to sell them at the rate of 2 for 3 cents. He made a profit of $1.26 on the entire transaction. How many dozen apples did he buy?

20. A baseball team has won 30 games and lost 24. It has 16 games yet to play. How many of these may it lose and yet win 60 per cent of the total?

21. A student has an average grade of 86 for four subjects. What grade must she make in a fifth subject so that her average will be 88?

22. A can do a piece of work in 3 days, B in 4 days, and C in 5 days. How long will it take them, working together, to do the same work?

Solution. By the conditions of the problem A does $\frac{1}{3}$ of the work in one day, B does $\frac{1}{4}$ of the work in one day, and C does $\frac{1}{5}$ of the work in one day. Let x represent the number of days required by A, B, and C together to do the work.

Then let $\frac{1}{x}$ = the fractional part of the work the three together do in one day.

Therefore $$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{1}{x}, \text{ etc.}$$

Solving, $$x=\frac{60}{47}.$$

Check as usual.

23. A can do a piece of work in 6 days and B in 5 days. How many days will they require, working together?

24. A can do a piece of work in 6 days, B and C each in 8 days, D in 15 days. How many days will they require, working together.

25. A can do a piece of work in 6 days, B in $4\frac{1}{2}$ days. How many days will they require, working together?

26. A can do a piece of work in 9 days, A and B, working together, in $4\frac{4}{5}$ days. How long would it take B alone?

27. A can do a piece of work in $5\frac{1}{4}$ days, B in $4\frac{1}{5}$ days, A, B, and C together in $1\frac{2}{5}$ days. How long would it take C alone?

28. A can do a piece of work in 8 days. After he has worked 3 days, B joins him and they finish the work in 2 more days. How long would it have taken B to do the work alone?

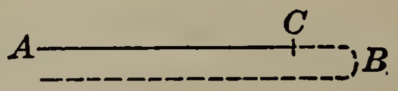
HINT. What fractional part of the work does A do in 1 day? in 3 days? in 5 days? What fractional part does B do in 1 day? in 2 days?

29. A can do a piece of work in 4 days, B in 6 days. After A has worked alone for 1 day, B joins him and they finish the job together. How long does it take them to complete the work?

30. A mass of lead and tin weighing 64 pounds contains 12 pounds of tin. How many pounds of tin must be added so that 50 pounds of the mixture will contain 15 pounds of tin?

31. Two motorists start at the same time to ride from A to B, 240 miles distant. One travels 10 miles an hour more than the other. The faster motorist reaches B and at once starts back, meeting the slower one at C, 192 miles from A. Find the rate of each.

Solution. The problem states that the two travel at different rates, that they travel different distances, but that the time is the same for each. Hence the equation must be formed by expressing the time t, or $\frac{d}{r}$, for each and equating the two expressions for t.



The two men together cover twice the distance from A to B, or 480 miles. As the slower one travels 192 miles, the faster travels 480 − 192, or 288 miles. If x equals the rate of the slower motorist in miles per hour, we have:

	d IN MILES	r IN MILES PER HOUR	$\frac{d}{r} = t$ IN HOURS
Slower motorist	192	x	$\frac{192}{x}$
Faster motorist	$480 - 192 = 288$	$x + 10$	$\frac{288}{x+10}$

Hence $$\frac{192}{x} = \frac{288}{x+10}.$$

Solving, we obtain $x = 20$, the rate of the slower motorist in miles per hour, and $x + 10 = 30$, the rate of the faster motorist.

Check. $\frac{192}{20} = 9.6$ and $\frac{288}{30} = 9.6$.

32. Two motorists, A and B, start at the same time to ride from X to Y, 180 miles distant. A travels 8 miles per hour less than B. The latter reaches Y and at once turns back, meeting A 20 miles from Y. Find the rate of each.

33. A train runs 280 miles. On the return trip it increases its rate by 5 miles an hour and makes the run in an hour less time. Find the rates going and returning.

34. An automobile makes a run of 120 miles. The chauffeur then increases the speed by 10 miles an hour and returns over the same route in 2 hours less time. Find the rates, going and returning.

35. A bicyclist traveling 18 miles per hour was overtaken $11\frac{1}{3}$ hours after he started by an automobile which left the same starting point 3 hours and 20 minutes later. What was the rate of the automobile?

36. An automobile makes a run of 300 miles. On the return trip the chauffeur decreases the speed by 5 miles an hour and requires 5 hours longer to cover the distance. Find the speed each way.

37. A man travels at a uniform rate from A to B, 150 miles distant. He travels the first 90 miles without stopping. The rest of the journey, including a delay of 3 hours, takes the same time as the first part. Find his speed.

HINTS. By reading the problem we discover that the distances covered in the first and second portions of the journey are different, that the time of travel is not the same for each, but that the rate throughout is the same. Hence one should find the two expressions for the rate r, or $\frac{d}{t}$, and set them equal to each other.

38. A leaves a certain point and walks at the rate of $3\frac{1}{2}$ miles per hour. Two and a half hours later B leaves the same point and drives in the opposite direction at the rate of 12 miles per hour. How much time must elapse after A starts before they will be 100 miles apart?

39. A and B start at the same time from two points 360 miles apart and travel toward each other. A's rate is 3 miles per hour less than B's. The latter, having been delayed 2 hours on the way, has traveled the same distance as A when they meet. Find the speed of each.

40. A man rows 4 miles per hour in still water. He finds that it requires 5 hours to row upstream a certain distance and 3 hours to return. Find the speed of the current.

HINT. Let $x =$ the speed of the current. Then $4 - x =$ the speed of the boat upstream, and $4 + x =$ the speed downstream.

41. A man who can row $4\frac{1}{2}$ miles per hour in still water rows up a stream the rate of whose current is 2 miles per hour. After rowing back he finds that the entire trip took 8 hours. How far upstream did he go?

42. A man who can row $3\frac{3}{4}$ miles per hour in still water rows downstream and returns. The rate of the current is $1\frac{1}{4}$ miles per hour, and the time required for the round trip is 9 hours. How many hours did he take to return?

43. The first transatlantic non-stop flight was made on June 14–15, 1919, by Alcock and Brown from St. John's, Newfoundland, to Clifden, Ireland, a distance of 1960 miles. Had their speed been 21 miles per hour less, the time would have been 3.4 hours more. State the equation which expresses this condition.

44. A mixture contains 3 gallons of gasoline and 5 gallons of kerosene. How many gallons of kerosene must be added to make a mixture that is $\frac{3}{4}$ kerosene?

Solution. In the final mixture,

$$\frac{\text{kerosene}}{\text{kerosene} + \text{gasoline}} = \frac{3}{4}.$$

Let $x =$ gallons of kerosene to be added.

Then $5 + x =$ gallons of kerosene in final mixture,

and $8 + x =$ total gallons of final mixture.

But
$$\frac{5+x}{8+x}=\frac{3}{4}.$$
$$20+4x=24+3x,$$
and
$$x=4.$$

Check. Total gallons of final mixture $= 5 + 3 + 4 = 12$ gallons.

Kerosene in final mixture $= 5 + 4 = 9$ gallons.

$$\tfrac{9}{12}=\tfrac{3}{4}.$$

45. A cook wishes to make salad dressing consisting of $\frac{2}{3}$ olive oil and $\frac{1}{3}$ vinegar. She has a mixture of 1 cup of oil and 1 cup of vinegar. How much oil must she add to produce the required mixture?

46. How much milk must be added to a 5-quart mixture of equal parts of milk and water to give a resulting mixture which is $\frac{2}{3}$ milk?

47. A man has 5 gallons of gasoline and an unlimited supply of a mixture which is $\frac{1}{4}$ gasoline and $\frac{3}{4}$ kerosene. How much of this mixture must he add to his gasoline to produce a mixture which is half gasoline and half kerosene?

48. A painter wishes to make a mixture consisting of $\frac{1}{16}$ linseed oil and $\frac{15}{16}$ paint. He accidentally pours 1 quart of oil into a gallon of paint. How much paint must he add to produce a mixture in the proportion which he wishes?

CHAPTER XVIII

LINEAR SYSTEMS

102. Definitions. A simple or *linear* equation in one or more unknowns is one which may be put in such form that

(*a*) no unknown appears in any denominator;
(*b*) only one unknown appears in any term;
(*c*) only the first power of any unknown is involved.

The following equations are linear: $5x - 2y = 3$; $2m + 3r - 2t = 0$. The following equations are not linear: $2x + 5xy - 7y = 2$; $\frac{4}{u} - \frac{6}{v} + \frac{2}{w} = 2$; $x^2 + 2x + 5y - 3 = (2x - 1)(5y + 8)$.

In a previous chapter we have seen that a simple equation in one unknown has only one root. In other words, the value of the unknown in such an equation is a constant.

Thus the value of the unknown in the equation $5x + 3 = 8$ is the number 1, and this is the only root of the equation.

A linear equation in two unknowns is satisfied by an unlimited number of pairs of values of the two unknowns. A single equation in more than one unknown is called an *indeterminate* equation. The unknowns are sometimes called variables.

Thus, the equation $x + y = 7$ is satisfied by any pair of numbers whose sum is 7. Evidently, if one includes positive, negative, and fractional numbers, there is no limit to the number of pairs whose sum is 7. (See page 276.)

Two or more equations involving two or more unknowns are called a *system* of equations.

A system of two equations both of which are satisfied by the same values of the unknowns is called a *simultaneous system* of equations.

In order that the two equations $x + y = 9$ and $x - y = 1$ may form a simultaneous system, the two numbers that satisfy both equations must be such that their sum is 9 while their difference is 1. These conditions are satisfied by $x = 5$, $y = 4$.

A *set* of values (one for each unknown) which satisfies an equation in two or more unknowns is sometimes called a solution of the equation; and a set which satisfies a system is often called a solution of the system. In this book, however, the word *solution* will be used to denote the *process of solving* either a single equation or a system. The values of the unknown which satisfy an equation in one unknown will be called *roots*, and a set of values for the unknowns satisfying an equation in two or more unknowns, or a system of such equations, will be called a *set of roots*.

ORAL EXERCISES

In Exercises 1–3 find the value of x corresponding to each of the values for y indicated at the right, and in Exercises 4–6 the values of y corresponding to the given values of x:

1. $x + y = 0$. $y = 0, 1, 3$.
2. $x + 2y = 3$. $y = -1, 3, 6$.
3. $x - y = 1$. $y = 2, -5, 0$.
4. $2x - y = 4$. $x = -3, 6, 5$.
5. $x + 6y = 5$. $x = 4, 2, -1$.
6. $3x - 2y = 6$. $x = 3, -9, 7$.

In Exercises 7–12 determine which of the pairs of numbers written at the right of each equation satisfies that equation. (The first number of a pair always denotes the value of x and the second number the value of y.)

7. $5x - 2y = 3$. (7, 6); (1, 1); (4, 2).
8. $x + y = 4$. (2, 5); (6, 8); (1, 3).
9. $2x - 5y = 2$. (5, 2); (6, 2); (8, 1).
10. $x + 2y = 0$. (1, 2); (5, − 1); (4, − 2).
11. $2x - y = 3$. (2, 1); (7, 6); (3, 2).
12. $2x + 4y = 8$. (3, 4); (2, 1); (6, − 1).

In Exercises 13–16 find two pairs of numbers which satisfy each equation, and two other pairs which do not.

13. $3x + 4y = 6$.
14. $2x - 5y = 1$.
15. $3x - y = 0$.
16. $x + 9y = -3$.

103. Solution by addition and subtraction. It was shown in the previous section that there is an unlimited number of sets of roots of a given linear equation in x and y, and that there is also an unlimited number of pairs of values of x and y which do not satisfy the equation. We now proceed to give an algebraic method of determining whether or not there is any common set of roots for two given linear equations. It turns out that there is usually one and only one such set of roots. This set may always be found by the method illustrated in the following

EXAMPLES

1. Solve the system $\begin{cases} 2x + y = 4, & (1) \\ 2x - 3y = -4. & (2) \end{cases}$

Solution. Eliminate x first, thus:

(1) − (2), $4y = 8$. (3)

(3) ÷ 4, $y = 2$. (4)

Substituting 2 for y in either (1) or (2), say in (1),

$$2x + 2 = 4. \qquad (5)$$

Solving (5), $$x = 1. \qquad (6)$$

Check. Substituting 1 for x and 2 for y in (1) and (2) gives the identities

$$2 + 2 = 4$$

and $$2 - 6 = -4.$$

2. Solve the system $\begin{cases} 4x + 3y = 0, & (1) \\ 10x - 7y = 58. & (2) \end{cases}$

Solution. Eliminate x first, thus:

(1) · 5, $$20x + 15y = 0. \qquad (3)$$

(2) · 2, $$20x - 14y = 116. \qquad (4)$$

(3) − (4), $$29y = -116. \qquad (5)$$

(5) ÷ 29, $$y = -4. \qquad (6)$$

Substituting − 4 for y in (2),

$$10x + 28 = 58. \qquad (7)$$

Solving (7), $$x = 3. \qquad (8)$$

Check. Substituting 3 for x and − 4 for y in (1) and (2) gives

$$12 - 12 = 0$$

and $$30 + 28 = 58.$$

Either x or y could have been eliminated first. The multipliers necessary to eliminate y are 7 and 3; the multipliers necessary to eliminate x are 5 and 2.

Since the object of solving a system of equations in x and y is the discovery of a set of values for x and y which will satisfy both equations at the same time, we are therefore safe in saying that x denotes the same number in both equations (1) and (2). The fact that the x's disappear when equation (2) is subtracted from equation (1) in Example 1 does not depend on the fact that the x's look alike, but solely upon the fact that the value represented by x has been eliminated from the first members of (1) and (2), hence a resulting equation is found in which x does not appear. The line of reasoning presented above is equally true for the unknown y.

When the notation (3) – (4) is used in a solution, as illustrated in Example 2, page 255, it indicates the subtraction of the first member of equation (4) from the first member of equation (3), the subtraction of the second member of equation (4) from the second member of equation (3), and the writing of the two results as an equation. The process of adding the corresponding members of the two equations is indicated by writing (3) + (4).

The notation (1) · 5 indicates that both members of equation (1) are multiplied by 5, and (5) ÷ 29 indicates that both members of (5) are divided by 29.

With the meanings just explained it is customary to speak of the addition or the subtraction of two equations and of the multiplication or division of an equation by a number.

The preceding method of solving a system of equations is summarized in the

RULE. *If necessary, multiply the first equation by a number, and the second equation by another number, such that the coefficients of the same unknown in each of the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add. Then solve the equation thus obtained.

Substitute the value just found, in the simplest of the preceding equations which contain both unknowns, and solve for the other unknown.

CHECK. *Substitute for each unknown in the original equations its value as found by the rule. If the resulting equations are not obvious identities, simplify them until they become such.*

An attempt to solve by the rule the pair

$$\begin{cases} 3x - 6y = 40, & (1) \\ x - 2y = 8, & (2) \end{cases}$$

gives

$$3x - 6y = 40, \qquad (3)$$

$$3x - 6y = 24. \qquad (4)$$

(3) – (4), $0 = 16$, which is false.

This result indicates that (1) and (2) do not form a simultaneous system but are *incompatible* equations.

An attempt to solve by the rule the system $\begin{cases} x + 2y = 8, \\ 3x + 6y = 24, \end{cases}$ gives $0 = 0$. Here each member of the second equation is just three times the corresponding member of the first equation. If we choose to regard the two equations as really different, which is not at all necessary, we say that they have an infinite (unlimited) number of sets of roots. Two or more equations having this property constitute an *indeterminate system.*

EXERCISES

Solve the following systems and check the sets of roots obtained:

1. $\begin{matrix} x + y = 6, \\ x - y = 2. \end{matrix}$

2. $\begin{matrix} x + y = -1, \\ x - y = 5. \end{matrix}$

3. $\begin{matrix} x + 2y = 7, \\ x - y = -2. \end{matrix}$

4. $\begin{matrix} x + y = 8, \\ x + 3y = 12. \end{matrix}$

5. $\begin{matrix} x - y = 1, \\ 2x + y = 14. \end{matrix}$

6. $\begin{matrix} 3x + 2y = 38, \\ 2x - y = 23. \end{matrix}$

7. $\begin{matrix} x + 5y = 9, \\ 2x + 4y = 6. \end{matrix}$

8. $\begin{matrix} 2y + x = -3, \\ x + y = -1. \end{matrix}$

9. $\begin{matrix} 3x + 5y = -5, \\ x + y = -1. \end{matrix}$

10. $\begin{matrix} 5x + 4y = 10, \\ 6x - 3y = -27. \end{matrix}$

11. $\begin{array}{l} 3x + 2y = -12, \\ 2x + 5y = -19. \end{array}$

12. $\begin{array}{l} 3x - y = -1, \\ 12x + 11y = 56. \end{array}$

13. $\begin{array}{l} 9m + 5p = 109, \\ 7m - 10p = 57. \end{array}$

14. $\begin{array}{l} 3r - 2s = 12, \\ 4r + 6s = 68. \end{array}$

15. $\begin{array}{l} 6p + 5q = 60, \\ 12p + 13q = 138. \end{array}$

16. $\begin{array}{l} t + 10u = 57, \\ 3t - 5u = -39. \end{array}$

17. $\begin{array}{l} 10x + 7y = 27, \\ 4x - 9y = -1. \end{array}$

18. $\begin{array}{l} 13x + 7y = -79, \\ 12x - 25y = -10. \end{array}$

19. $\begin{array}{l} 5x - 7y = 78, \\ 3x + 13y = -108. \end{array}$

20. $\begin{array}{l} 2x - 5z = -27, \\ 7x + 13z = -3. \end{array}$

21. $\begin{array}{l} 13x - 5y = -14, \\ 5x + 3y = 34. \end{array}$

22. $\begin{array}{l} 2m + 5n = 15, \\ m - 4n = 1. \end{array}$

104. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the following

EXAMPLE

Solve the system $\begin{cases} x + 5y = -11, & (1) \\ 2x - 3y = 17. & (2) \end{cases}$

Solution. From (1), $x = -5y - 11. \quad (3)$

Substituting $-5y - 11$ for x in (2),

$$2(-5y - 11) - 3y = 17. \tag{4}$$

Simplifying, $-10y - 22 - 3y = 17. \quad (5)$

Collecting, $-13y = 39, \quad (6)$

or, $y = -3. \quad (7)$

Substituting -3 for y in (3),

$$x = 15 - 11 = 4. \tag{8}$$

Check. Substituting 4 for x and -3 for y in (1) and (2) gives

$$4 - 15 = -11$$

and $$8 + 9 = 17.$$

The method of the preceding solution for solving a system of two linear equations is stated in the

Rule. *Solve either equation for one unknown in terms of the other.*

Substitute this value in place of the unknown in the equation from which it was not obtained, and solve the resulting equation.

Substitute the definite value just found in the simplest of the preceding equations which contain both unknowns, and solve, thus obtaining a definite value for the other unknown.

Check. *See page 256.*

The method of substitution emphasizes the fact that the values of x and y which are sought are the same in both equations. Hence an expression for an unknown obtained from one equation is substituted for that unknown in the other equation. This method is useful when one of the unknowns can be expressed in terms of the other without fractions or when simple fractions only are involved.

EXERCISES

Solve by the method of substitution and check the sets of roots obtained:

1. $m - 2n = 1,\quad 2m - n = 5.$

3. $9s + 10t = 113,\quad 4s - 3t = 13.$

5. $3x + 5y = 61,\quad 10x - y = 62.$

2. $x + 3y = 14,\quad 3x - y = 2.$

4. $y + z = 1,\quad 3y - 10z = -36.$

6. $3s + 2t = 7,\quad s + t = -1.$

7. $x + 11y = -42,\quad x - 7y = 30.$

10. $43w + 10s = 189,\quad 5w - 3s = -3.$

8. $5x = 2y + 3,\quad 3y - 6x = -3.$

11. $2x - 19y = -212,\quad 3y + 7x = 92.$

9. $2x - 7y = 37,\quad 2x + y = -3.$

12. $3x + 2y = 1,\quad 5x - 4y = 3.$

105. Simultaneous equations containing fractions. The method of solving a system of two linear equations containing fractions is given in the following

EXAMPLE

Solve the system
$$\begin{cases} \dfrac{2x}{5}+\dfrac{3y}{2}=\dfrac{34}{5}, & (1)\\[2mm] \dfrac{x}{3}+\dfrac{3y}{7}=\dfrac{50}{21}. & (2)\end{cases}$$

HINTS. (1) · 10, $4x+15y=68.$ (3)

(2) · 21, $7x+9y=50.$ (4)

The system (3) and (4) can now be solved either by addition and subtraction or by substitution.

As in the foregoing solution, it is usually best to clear the equations of fractions and write them in the form of (3) and (4) before attempting to eliminate one of the unknowns. Equations (3) and (4) are each in what is known as the *general form* of a linear equation in two unknowns. This form is represented for all such equations by $ax+by=c$, where a, b, and c denote numbers, or known literal expressions.

EXERCISES

Solve the following systems and check the sets of roots obtained:

1. $\begin{cases} 2x+\dfrac{3y}{2}=\dfrac{39}{2}, \\[2mm] x-\dfrac{y}{2}=\dfrac{27}{2}.\end{cases}$

2. $\begin{cases} \dfrac{3x}{7}-2y=-\dfrac{47}{7}, \\[2mm] \dfrac{5y}{4}+\dfrac{3x}{2}=\dfrac{19}{2}.\end{cases}$

3. $$\begin{cases} \dfrac{x+y}{7}+\dfrac{x-y}{5}=\dfrac{18}{35}, \\ \dfrac{x+y}{3}-\dfrac{x-y}{4}=\dfrac{23}{12}. \end{cases}$$

4. $$\begin{cases} \dfrac{2x}{5}+\dfrac{y}{4}=\dfrac{23}{20}, \\ \dfrac{9x}{4}+y=\dfrac{21}{4}. \end{cases}$$

5. $$\begin{cases} \dfrac{x+y}{3}+\dfrac{x-y}{5}=\dfrac{6}{5}, \\ \dfrac{2x-2y}{7}-\dfrac{3x-3y}{4}=-\dfrac{13}{28}. \end{cases}$$

6. $$\begin{cases} \dfrac{4x+3y}{6}-\dfrac{2x+5y}{2}=\dfrac{35}{6}, \\ \dfrac{x+y}{2}+\dfrac{x-y}{3}=-\dfrac{1}{12}. \end{cases}$$

7. $$\begin{cases} \dfrac{m+2n}{5}-\dfrac{2n}{3}=\dfrac{11}{15}, \\ \dfrac{4m-n}{2}+m=4. \end{cases}$$

11. $$\begin{cases} x+\dfrac{5y}{3}=\dfrac{5}{4}, \\ \dfrac{7x}{3}-\dfrac{2y}{9}=-\dfrac{43}{36}. \end{cases}$$

8. $$\begin{cases} \dfrac{x+y}{1}+\dfrac{2x-2y}{3}=\dfrac{4}{3}, \\ \dfrac{x-y}{3}-\dfrac{x+y}{7}=\dfrac{2}{3}. \end{cases}$$

12. $$\begin{cases} \dfrac{2a-5b}{3}+\dfrac{2a}{5}=\dfrac{22}{45}, \\ \dfrac{4a+2b-3}{13}+a=\dfrac{11}{6}. \end{cases}$$

9. $$\begin{cases} \dfrac{r+s}{2}+\dfrac{r-s}{3}=\dfrac{2}{3}, \\ \dfrac{2r-3s}{5}+2r=3. \end{cases}$$

13. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{6}{5}, \\ \dfrac{1}{x}-\dfrac{1}{y}=-\dfrac{4}{5}. \end{cases}$$

10. $$\begin{cases} \dfrac{l+m-2}{3}+\dfrac{m}{2}=-\dfrac{10}{3}, \\ \dfrac{2m-l+3}{2}-\dfrac{5m+1}{3}=\dfrac{1}{2}. \end{cases}$$

HINT. Solve without clearing of fractions, using addition and subtraction.

14. $$\begin{cases} \dfrac{2}{x}+\dfrac{2}{y}=3, \\ \dfrac{3}{x}-\dfrac{5}{y}=\dfrac{1}{2}. \end{cases}$$

15. $$\begin{cases} \dfrac{2}{m}+\dfrac{3}{n}=1, \\ \dfrac{4}{n}-\dfrac{5}{m}=-\dfrac{19}{3}. \end{cases}$$

16. $$\begin{cases} 7r+\dfrac{3}{t}=-6, \\ 2r+\dfrac{5}{t}=-\dfrac{1}{3}. \end{cases}$$

17. $$\begin{cases} \dfrac{5}{K}+2L=\dfrac{7}{2}, \\ \dfrac{6}{K}-5L=-18. \end{cases}$$

18. $$\begin{cases} \dfrac{x+y-5}{2}+\dfrac{x-y}{3}=-\dfrac{1}{3}, \\ \dfrac{x-y+2}{6}-\dfrac{x+y+1}{3}=\dfrac{1}{2}. \end{cases}$$

19. $$\begin{cases} \dfrac{x+z-1}{5}+z-2=\dfrac{21}{5}, \\ \dfrac{2x-z}{3}+\dfrac{z+x}{4}=\dfrac{17}{12}. \end{cases}$$

20. $$\begin{cases} \dfrac{5}{x+y}+\dfrac{7}{3x-5y+4}=-2, \\ x-\dfrac{5y}{3}=\dfrac{10}{3}. \end{cases}$$

21. $$\begin{cases} \dfrac{9x-y}{7x+2y}=\dfrac{3}{4}, \\ 2x-5y=-22. \end{cases}$$

22. $$\begin{cases} \dfrac{5x}{2x-5y+3}=-1, \\ \dfrac{x}{2}+\dfrac{y}{1}=\dfrac{3}{2}. \end{cases}$$

23. $$\begin{cases} \dfrac{2x-4y}{3}=\dfrac{9x+10y+72}{5}, \\ 2x-\dfrac{5y}{6}=\dfrac{49}{6}. \end{cases}$$

24. $$\begin{cases} \dfrac{4x-3y+2}{5}=2y+\dfrac{49}{5}, \\ \dfrac{2x}{3}+\dfrac{5y}{2}=-\dfrac{37}{6}. \end{cases}$$

25. $$\begin{cases} \dfrac{3x}{2}+\dfrac{y}{3}=\dfrac{11}{3}, \\ \dfrac{4x}{5}+\dfrac{2y}{7}=\dfrac{76}{35}. \end{cases}$$

26. $$\begin{cases} \dfrac{5}{m+n}+\dfrac{2}{m-n}=\dfrac{9}{4}, \\ \dfrac{9}{m-n}+\dfrac{10}{m+n}=7. \end{cases}$$

27. $$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=-\dfrac{1}{2}, \\ \dfrac{5}{x}+\dfrac{2}{3y}=-\dfrac{14}{3}. \end{cases}$$

28. $$\begin{cases} \dfrac{2}{3p}-\dfrac{5}{-2n}=-\dfrac{43}{9}, \\ \dfrac{7}{2p}+\dfrac{10}{3n}=-\dfrac{11}{2}. \end{cases}$$

EXERCISES

Solve the following systems involving decimals, and check the sets of roots found:

1. $\begin{cases} 2m + .3n = .82, \\ n - 4m = -2.6. \end{cases}$

2. $\begin{cases} 2x + 5y = 43, \\ 3x - .2y = 10.6. \end{cases}$

3. $\begin{cases} .13r - .24s = -.545, \\ .5r + .61s = .97. \end{cases}$

4. $\begin{cases} .3x - .5y = .4, \\ .2x + .4y = 1. \end{cases}$

5. $\begin{cases} .2x + .3y = .75, \\ .5x + .6y = 1.8. \end{cases}$

6. $\begin{cases} .8x + .2y = .7, \\ .4x - 2.0y = 10.15. \end{cases}$

7. $\begin{cases} .06x + .67y = 4.32, \\ .9x - 2y = -7.5. \end{cases}$

8. $\begin{cases} .8x + .33y = -2.4, \\ .1x - .66y = -.2. \end{cases}$

9. $\begin{cases} 4x + 1.5y = 9.6, \\ 5x - 3.5 + 4y = 19.125. \end{cases}$

10. $\begin{cases} 3x + 13y = 74, \\ 12x - 5.3y = -47.8. \end{cases}$

11. $\begin{cases} .2x + 5y = 15.6, \\ 3x - 2.5y = -6.25. \end{cases}$

12. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 6, \\ \dfrac{1}{x} - \dfrac{1}{y} = -\dfrac{1}{.5}. \end{cases}$

13. $\begin{cases} \dfrac{5}{x} + \dfrac{6}{y} = 4, \\ \dfrac{7}{x} + \dfrac{1}{y} = 1.9. \end{cases}$

14. $\begin{cases} \dfrac{.2}{x} - \dfrac{.3}{y} = \dfrac{1.6}{6}, \\ \dfrac{5}{x} + \dfrac{.5}{y} = \dfrac{1}{3}. \end{cases}$

15. $\begin{cases} \dfrac{2}{x} + 10y = -39.6, \\ \dfrac{3}{x} - 5y = 20.6. \end{cases}$

16. $\begin{cases} 25x + .25y + \dfrac{117}{8} = 4(x+y), \\ \dfrac{x-y}{5} + 1.8 = \dfrac{y}{2}. \end{cases}$

17. $\begin{cases} 2x + \dfrac{.5}{y} = 5.5, \\ \dfrac{3.2}{y} - .4x = -4.4. \end{cases}$

18. $\begin{cases} 2x - 3y = -1, \\ 5.5x + 3.2y = 8.7. \end{cases}$

19. $\begin{cases} 15a - 2.5b = 4.75, \\ .3a - .4b = .08. \end{cases}$

20. $\begin{cases} \dfrac{.2}{x} - \dfrac{.5}{y} = 2, \\ \dfrac{.3}{x} + \dfrac{.1}{y} = 4.7. \end{cases}$

In the following problems the student should state *two* equations in *two* unknowns. Instead of using x and y, the *first letter* of the word denoting an unknown should be used to represent that unknown wherever possible. Thus in Problem 7, p. 265, d should represent the number of dimes and q the number of quarters. The plan here suggested is desirable for many reasons, and should be followed in all problems containing two or more unknowns unless the words denoting two of the unknowns begin with the same letter.

In solving problems like 1–39 on pages 175–179 there are really two unknowns involved; but one of the equations to which each of those problems leads is so simple that what amounts to the method of substitution was employed by expressing one unknown in terms of the other.

EXAMPLE

The difference between two numbers is 17, and their sum is 33. Find the numbers.

Solution. Let $x =$ one number,

and $y =$ the other number.

Then $x - y = 17, \quad (1)$

and $x + y = 33. \quad (2)$

Solving for x and y, we get

$$x = 25,$$
$$y = 8.$$

PROBLEMS

1. The difference between two numbers is 25 and their sum is 41. Find the numbers.

2. The difference between two numbers is 3. Their sum is 11. Find the numbers.

3. What are the two numbers whose sum is 10 and whose difference is 6?

4. Find the two numbers whose sum is 5 and whose difference is 25.

5. The value of a certain fraction is $\frac{1}{2}$ when 1 is added to the numerator. When 4 is subtracted from the denominator the value becomes $\frac{3}{4}$. Find the fraction.

6. The value of a certain fraction is $\frac{1}{2}$. If 2 is added to the denominator and 1 is subtracted from the numerator the fraction becomes equal to $\frac{1}{3}$. What are the values of the numerator and denominator of the fraction?

7. The value of a collection of quarters and dimes containing 100 coins is $10.90. How many coins of each kind are there?

8. There are two kinds of coins in a collection. When the collection contains 8 of one kind and 5 of the other it is worth $1.65. If there are 10 of the first kind and 13 of the second kind, the lot amounts to $3.75. What are the denominations of the two types of coins?

9. The difference of the numerator and denominator of a certain fraction is 1. Subtracting 5 from the numerator and adding 4 to the denominator makes the fraction equal to $\frac{1}{2}$. What is the original value of the fraction?

10. Two boys on a seesaw balance when one is 6 feet from the fulcrum and the other is 4 feet from the fulcrum. A third boy weighing 70 pounds joins the first boy, and the two balance the second boy in his original position when they are $2\frac{1}{2}$ feet from the fulcrum. What are the weights of the first and second boys?

11. The sum of two weights is 16 pounds. They balance each other when they are $\frac{1}{2}$ foot and $1\frac{1}{2}$ feet from the fulcrum respectively. Find the weights.

12. A man has $5000 invested, partly in a savings bank paying 4 % interest and partly in bonds paying 6 %. His income from the two sources is $280 a year. How much has he invested in each place?

13. Part of $15,000 is invested at 5 % and the remainder at 4 %. The income from the 4 % investment is twice that from the 5 %. How much is invested at each rate of interest?

14. A is now $\frac{2}{3}$ as old as B. Five years ago B was twice as old as A. What are their ages now?

15. In 10 years C will be $\frac{4}{3}$ as old as D. In 5 years D will be $\frac{5}{7}$ as old as C. What is the present age of each?

16. A man has $2000 to invest, part at 7 % and the remainder at 4 %. He wishes an income of $125 per year from this money. How much must he invest at each figure?

17. The sum of the digits of a two-digit number is 7. The number plus 27 is the original number with digits reversed. Find the original number.

Solution. Let $t =$ the digit in tens' place,

and $u =$ the digit in units' place.

Then $t + u = 7$. (1)

But t standing in tens' place has *its numerical value multiplied by* 10. Therefore the number is represented by the binomial $10\,t + u$, and the number formed by the digits in reverse order is represented by the binomial $10\,u + t$.

Hence $10\,t + u + 27 = 10\,u + t$. (2)

Simplifying (2), $u - t = 3$. (3)

Solving (1) and (3), $t = 2$ and $u = 5$.

Hence the number is 25.

18. The sum of the digits of a two-digit number is 6. The digits are reversed if 36 is added to the number. What is the original number?

19. In a certain two-digit number the tens' figure is twice the units' figure. If the number with digits reversed is subtracted from the original number the remainder is 18. Find the original number.

20. A certain number divided by $\frac{1}{2}$ the sum of the digits gives 17. If 36 is subtracted from the number, the remainder is the original number with digits reversed. Find the original number.

21. A certain two-digit number plus 8 is equal to ten times one less than the sum of the digits. The tens' digit is twice the units' digit. What is the original number?

22. The sum of a certain two-digit number and the sum of the digits of the number is 15. The number minus the sum of the digits is 9. Find the number.

The *reciprocal* of a number is a fraction of which the numerator is 1 and the denominator is the number itself. Thus $\frac{1}{2}$ and $\frac{1}{a}$ are the reciprocals of 2 and a respectively.

23. What are the reciprocals of 3, 5, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, 1, $1\frac{1}{2}$, $8\frac{7}{10}$?

24. The sum of the reciprocals of two numbers is $\frac{7}{10}$, and the difference of the reciprocals is $\frac{3}{10}$. Find the numbers.

25. The sum of the reciprocals of two numbers is $\frac{7}{12}$. The greater reciprocal over the lesser reciprocal is equivalent to $1\frac{1}{3}$. Find the numbers.

26. A gives B \$50. B then has four times as much as A. Then B gives A \$100 and has left twice as much as A. How much money did each have in the first place?

27. A man has two barrels of gasoline. He takes 10 gallons from the first and puts it in the second. The second then contains three times as much as the first. Next he takes 30 gallons from the second and puts it in the first. The second barrel then contains $\frac{6}{10}$ as much as the first barrel. How much gasoline was there originally in each barrel?

28. A man walks in a railway train in the direction of the motion of the train for 2 minutes. He has then traveled with respect to the ground $\frac{13}{30}$ of a mile. He returns through the train at the same speed to his starting point, and has traveled in returning $\frac{7}{30}$ of a mile with respect to the ground. What was his rate of walking in miles per hour, and how fast was the train moving?

29. A man rows a boat 5 miles downstream in 1 hour and returns in $1\frac{1}{2}$ hours. What is his speed? How fast is the current in the river flowing?

30. A man travels 100 miles, 50 miles by train and 50 miles by autobus. He makes the trip in 2 hours and 40 minutes. On the return trip he travels 30 miles by train and 70 miles by autobus, and takes 2 hours and 56 minutes. What was the speed of the train? of the autobus?

31. A man must make a trip of 200 miles. He can go part way by auto and the rest of the way by train. If he takes a bus going 20 miles per hour to the station and takes the train the remainder of the distance he can make the trip in $5\frac{5}{8}$ hours. If he takes a taxi traveling 30 miles per hour to the same station and the train for the remainder of the way he can make his journey in $5\frac{5}{24}$ hours. What is the distance to the station? How far does he go by train? What is the speed of the train?

32. In Problem 31 if the train fare is 4¢ per mile, the bus fare 5¢ per mile, the taxi fare 40¢ per mile, and the man's time is worth $5 per hour, what is the most economical way for him to travel? How much would his time have to be worth to make it economical for him to take the taxi to the train?

33. Two men, A and B, can do a piece of work together in 4 days. They work together on the job for $1\frac{1}{2}$ days and then B finishes it alone in $7\frac{1}{2}$ days more. How many days would each man require to do the work alone?

34. A boat going 18 miles per hour travels downstream from A to B in 2 hours. It then returns to a point, C, 7 miles below A, in 3 hours. What is the distance from A to B? from B to C? What is the speed of the current?

106. Literal equations in two unknowns. Linear systems in which the unknowns have literal coefficients are usually solved by the method of addition and subtraction.

EXERCISES

In the following exercises consider *a*, *b*, *c*, *d*, and *l* as known numbers; solve for the other letters involved and check:

1. $$\begin{gathered} x + y = 4a, \\ x - 2y = a. \end{gathered}$$

2. $$\begin{gathered} x + y = 5a, \\ x - y = a. \end{gathered}$$

3. $$\begin{gathered} m + n = c, \\ m - n = d. \end{gathered}$$

4. $$\begin{gathered} 3x + 4y = b, \\ 5x + 3y = c. \end{gathered}$$

5. $$\begin{gathered} 2x + 3cy = 3c, \\ y - x = -\frac{3c}{2}. \end{gathered}$$

6. $$\begin{gathered} \frac{2x}{3} + \frac{9y}{d} = 18, \\ 3dx - \frac{2y}{d} = -4. \end{gathered}$$

7. $$\begin{gathered} \frac{9x}{4} + \frac{3y}{5} = \frac{123a}{20}, \\ \frac{2x}{a} + \frac{5y}{2a} = \frac{7}{2}. \end{gathered}$$

8. $\frac{2x}{3} - \frac{5y}{6} = a,$
$\frac{x}{2} + \frac{y}{4} = -a.$

9. $0.25x + 0.6y = .45a,$
$0.5y - 0.4x = 0.01a.$

10. $\frac{2v}{5} + \frac{5s}{3} = -4.2a,$
$\frac{v}{2} - \frac{s}{3} = 2a.$

11. $4x + 3y = 22a,$
$5x + 7y = 47a.$

12. $xa + y = a,$
$\frac{x}{2} + \frac{y}{a} = 0.$

13. $ax + 2y = -\frac{4}{3}x - 3a^2,$
$3ay - 2x = 6(a^2 + a).$

14. $3m + 2n = (a + b),$
$12m + 9a = 4b + 18n.$

15. $x - 3y = 2b,$
$3x + 2y = 17b.$

16. $x + by = a,$
$2x - ay = \frac{a}{b}(4b + a).$

17. $\frac{1}{x} + \frac{1}{y} = a,$
$\frac{a}{x} - \frac{1}{y} = b.$

18. $\frac{a}{y} + \frac{b}{x} = d,$
$\frac{a}{y} - \frac{b}{x} = c.$

19. $cm + dn = d + c,$
$dn - cm = d - c.$

20. $ax - cy = b,$
$4ax - 3cy = a + 3b.$

21. $bz + cw = d,$
$lw + z = a.$

22. $.5m + .2n = .2,$
$1.2n - .3m = -2.76.$

23. $.2x - .3y = -.25l,$
$.5x + 1.3y = 1.65l.$

GENERAL ORAL EXERCISES

1. If 1 book costs $2a$ dollars, how much will d books cost?

2. If 3 books cost b dollars, how much will n books cost?

3. What is the perimeter of a rectangle of length l inches and breadth b inches? What is its area?

4. If n books cost d dollars, what will one book cost? b books?

5. The area of a rectangle is ab square inches. If the length is a feet, what is the width? the perimeter?

6. The length of a rectangle is a feet. The width is $b - a$ feet. What is the area? the perimeter?

7. A triangle has an altitude a feet and a base $b - l$ feet. What is the area?

8. A triangle has an altitude h feet and a base equal to $2\,b$ inches. What is its area?

9. A triangle has an altitude of l feet. If the altitude is $\frac{1}{2}$ the base in feet, what is the base? the area?

10. Two trains start from the same station and travel in opposite directions a and b miles per hour respectively. How far will each be from the station in t hours? How far apart will they be at that time?

11. If these trains travel in the same direction, how far apart will they be from each other in t hours? How far will each be from the station from which they started?

12. The sum of two numbers is 10, and the lesser is s. What is the greater? What is the difference? the product?

13. If x is A's age now, what does $(x - 5)$ indicate? What does the equation $x + 2 = 2(x - 5)$ mean?

14. A man receives d dollars per working day for 4 weeks. In this period he spends e dollars. How much money has he at the end of 4 weeks of 6 working days each?

15. A man can walk m miles in 1 hour. How far can he go in h hours? How long will it take him to walk d miles?

16. It costs a man d dollars a day to live. How long can he live on p dollars? How much money will it take to support n men one day? t days?

17. If 9 apples can be bought for c cents, how many can be had for d cents? How much will a apples cost?

18. A triangle has an area a square inches and base b feet. What is its altitude?

19. A farmer has fodder for 12 cattle n days. How long can m cattle live on the same amount of food?

20. One man does a piece of work in n days. How much can he do in d days? How long will it take m men to do the job if each man works at the same rate as the first man?

GENERAL PROBLEMS

1. The base of a triangle is b inches and the altitude is 6 inches. If the base is increased 3 inches, how much must the altitude be reduced to keep the area the same?

HINT. Let $x =$ the decrease of the altitude in inches.

Then $6b/2 = (b + 3)(6 - x)/2$, etc.

2. The altitude of a triangle is a inches and the base is b inches. If the base is increased by n inches, how much must the altitude be reduced to keep the original area?

3. A rectangle has a length of l inches and a breadth of b inches. If the length is reduced d inches, how much must the breadth be increased to keep the area the same?

4. The length and width of a rectangle are 6 and 4 inches respectively. The length is decreased by the same amount as the width is increased, while the area remains the same. How much is the length decreased?

5. The sum of two numbers is s and their difference is d. What are the numbers?

6. The sum of two numbers is q and their difference is one half of q. What are the numbers?

7. A certain number is 3 larger than another number. The quotient of the smaller divided by the larger is q. Find the numbers.

8. The value of a fraction is a. If 1 is added to the numerator the value becomes b. Find the numerator and the denominator.

9. If b is added to the numerator of a certain fraction, the resulting fraction is equal to 3. If a is added to the denominator the fraction is equal to 1. Find the numerator and the denominator.

10. If 2 is added to the numerator of a certain fraction, the resulting expression is equal to x. When 3 is subtracted from the denominator the fraction is equal to z. What are the values of the numerator and the denominator?

11. A collection of eggs contains a certain number costing $1 per dozen and another number costing 70¢ per dozen. Together the eggs cost b dollars. If the number of eggs of each grade had been interchanged, the cost of the whole would have been a dollars. How many dozen eggs were there of each grade?

12. A boy weighing p pounds balances a boy weighing w pounds on a seesaw. If the distance between the boys is l, what is the distance of each boy from the fulcrum?

13. A and B have together d dollars. A has m dollars more than B. How much has each?

14. If A gave g dollars to B, he would then have half as much as B. If B gave k dollars to A, he would have 1 dollar more than A. How much money has each?

15. Together A and B have $50. A gives g dollars to B and then B gives f dollars to A, whereupon they have equal amounts. How much money did each have?

16. A man has d dollars invested, part at 4% and part at 5% interest. His income is m dollars per year. How much money has he invested at each rate of interest?

17. Two men, A and B, start from points m miles apart and walk toward each other. A walks at r miles per hour, and B walks at s miles per hour. How long will they have to walk before they meet? How far will they be from their respective starting points when they meet each other?

18. A works half as fast as B. The two working together can do a piece of work in d days. How long would it take each man alone to do it?

19. A does a job in c days. A and B together do the job in d days. How many days would it take B to do the job alone?

20. A works four times as fast as B. The two together do a piece of work in p days. How long will it take each man alone to do the same job?

21. A and B together do a job in d days. If A works t times as fast as B, how long would it take each to do the work?

REVIEW EXERCISES

1. Solve $4(10 - 2x) - 3(x - 15) = 0$.

2. Solve $\dfrac{5x + 3}{4} = 7$.

3. Solve the system $\begin{cases} 2x + 5y = 24, \\ 3x - y = 2. \end{cases}$

4. Solve $\dfrac{3x - 5}{x - 3} + \dfrac{2x - 5}{x - 4} = \dfrac{35(x - 2)}{7x - 24}$.

5. Solve the system $\begin{cases} \dfrac{3x}{2} - \dfrac{5y}{3} + 2 = \dfrac{1}{6}, \\ \dfrac{2x}{3} + \dfrac{5y}{3} = \dfrac{17}{3}. \end{cases}$

Factor:

6. $a^2 + 2ab - 15b^2$. 7. $(a - b)^2 - x^2$. 8. $x^3 + 4x^2 + 4x$.

Reduce to a common denominator and simplify:

9. $\frac{1}{a-b} - \frac{1}{a+b}$.

10. $\frac{8}{2x-3} + \frac{5}{3-2x} - \frac{3x-4}{2x^2-x-3}$.

Divide:

11. $a^3 - a^2b + 2b^3$ by $a + b$.

12. $6x^3 + x^2 - 29x + 21$ by $2x - 3$.

Solve for x and then find its numerical value for the given values of the other letters:

13. $ax + x = m$, when $a = 2$ and $m = -3$.

14. $a(x - 1) - b = x - a$, when $a = 2$ and $b = 0$.

15. $ax + bx = m + x$, when $a = -4$, $b = \frac{1}{2}$, and $m = 4$.

16. $m(a + b - x) = n(a + b - x)$, when $a = -1$, $b = 2$, $m = -3$, and $n = 5$.

17. The sum of two numbers is 51. Their difference is 13. Find the two numbers.

18. Two boys weigh together 130 pounds. They balance on a seesaw when they are 3 feet and $3\frac{1}{2}$ feet from the fulcrum respectively. What are their weights?

19. A collection of nickels and quarters is worth \$2.40 and contains 20 coins. What is the number of each kind of coin in the collection?

20. A and B had between them \$1000. A spent $\frac{1}{2}$ of his share while B spent $\frac{1}{3}$ of his. If A spent \$25 more than B, what did each have in the first place?

CHAPTER XIX

GRAPHS OF EQUATIONS

107. Introduction. In Chapter II several devices were given for the graphic presentation of statistics. One of these, the line graph, is similar in its general features to the method employed for representing geometrically the relation between sets of numbers which are connected by an algebraic equation.

The question What two numbers added give seven? may be expressed by the equation $x + y = 7$. Here x and y are any two numbers whose sum is 7. It can be seen by inspection that if $x = 4$, $y = 3$; if $x = 1$, $y = 6$; if $x = 0$, $y = 7$; etc. We may also assign to x any value, say -2; then the equation becomes $-2 + y = 7$, whence $y = 9$. In this manner we may obtain an unlimited number of sets of related values for x and y, some of which are given in the following table:

$$x + y = 7$$

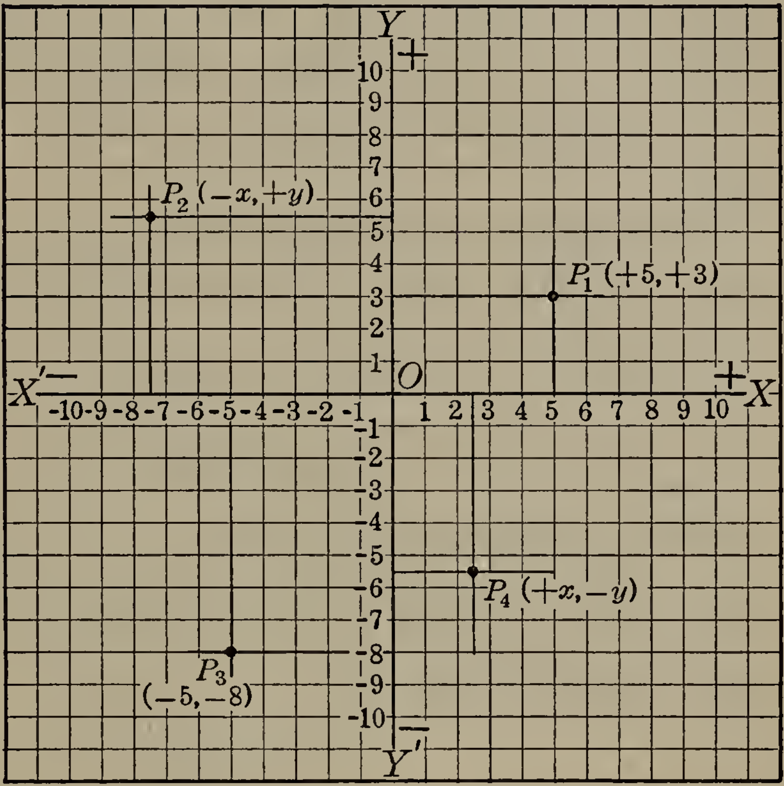
	A	B	C	D	E	F	G	H	I
If $x =$	5	1	0	6	7	-1	-2	8	9
then $y =$	2	6	7	1	0	8	9	-1	-2

108. Definitions and assumptions. In constructing the graph of an equation in two variables a number of assumptions must be made. These assumptions and some necessary definitions are now stated.

It is agreed:

I. To have two lines at right angles to each other, — as $X'OX$, called the *x-axis,* and $Y'OY$, called the *y-axis,* as in the figure below.

II. To have a line of definite length as a unit of distance. Then the number 2 will correspond to a distance twice the unit in length, the number $4\frac{1}{2}$ to a distance of $4\frac{1}{2}$ times the unit, etc.



III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the surface of the paper be the *x-distance* (or abscissa) of that point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the *y-distance* (or ordinate) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive* number, and the x-distance of a point to the *left* by a *negative* number; also that the y-distance of a point *above* the x-axis be represented by a *positive* number, and the y-distance of a point *below* the x-axis by a *negative* number. Briefly, *distances measured from the axes to the right or upward are positive, to the left or downward are negative.*

V. That every point in the surface of the paper corresponds to a *pair of numbers,* one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x-distance and the second the measure of the y-distance. Thus the point (5, 3), or P_1, in the figure, is the point whose x-distance is 5 and whose y-distance is 3. Again, the point $(-5, -8)$, or P_3, is the point whose x-distance is -5 and whose y-distance is -8.

The point of intersection of the axes is called the *origin.*

The values of the x-distance and the y-distance of a point are often called the *coördinates* of the point.

Though not an absolute necessity, cross-ruled paper is a great convenience in all graphical work. Excellent results, however, can be obtained with ordinary paper and a rule marked in inches and fractions of an inch for measuring distances. Hence the graphical work which follows should not be omitted even though it is found inconvenient to obtain cross-ruled paper for class use.

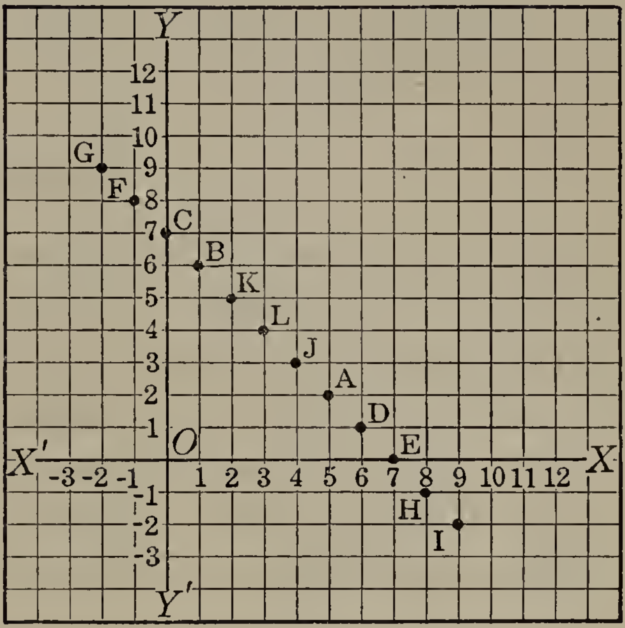
EXAMPLE

Using $\frac{1}{4}$ inch for the unit of measure, locate points corresponding to the coördinates which follow: A, (5, 2); B, (1, 6); C, (0, 7); D, (6, 1); E, (7, 0); F, $(-1, 8)$; G, $(-2, 9)$; H, $(8, -1)$; I, $(9, -2)$; J, (4, 3); K, (2, 5); and L, (3, 4).

Solution. According to VI, above, the x-distance of the point (5, 2) is 5 and its y-distance is 2. Hence to locate point A at (5, 2) we measure, according to IV, five units to the right of the origin on the x-axis and from that point two units upward.

To locate point F at $(-1, 8)$ we measure on the x-axis one unit to the left of the origin and from that point upward eight units. Point F thus located corresponds to $(-1, 8)$.

Points for the other pairs of numbers given in the example should be located by the pupil. The correct positions for these points can be seen in the figure.



Locating points as in the example above is called *plotting points.*

EXERCISES

Plot the following points with reference to the two axes. It is suggested that the unit of measure be $\frac{1}{2}$ inch.

1. $(5, 1)$; $(3, 4)$; $(3.25, 0)$; $(1, 2)$; and $(0, 0)$.

2. $(-1.5, 2)$; $(1, -3)$; and $(2.5, 4)$.

3. $(-2, -4)$; $(1.5, -0.5)$; $(-1, -1)$; and $(0, -3)$.

4. $(5, -2)$; $(-1, -3.5)$; $(2.25, -3)$; $(1.75, -6)$; and $(0.75, 3.25)$.

5. If the x-distance of a point is zero, where is the point located? Where is it located if both of its coördinates are zero?

109. The graph of an equation. On page 276 we computed several sets of values of x and y for the equation $x + y = 7$. Later these points were plotted in locating A, B, C, D, E, F, G, H, and I of the preceding example. It is evident from an inspection of their position that a straight line can be made to pass through all the points there located. The *line* drawn through these points is said to be the *graph of the equation* $x + y = 7$.

EXERCISES

1. Find and tabulate six pairs of values of x and y which satisfy the equation $x + 2y = 6$. Draw two axes and, using $\frac{1}{2}$ inch as the unit distance, plot each of the points. Are the six points in a straight line? Does $x = 3$, $y = 3$ satisfy this equation? Plot the point (3, 3). Is it on the graph of the equation? If the x- and y-distances of a point satisfy the equation $x + 2y = 6$, where is the point located? If the x- and y-distances of a point do not satisfy the equation $2x + y = 8$, what can be said of its location?

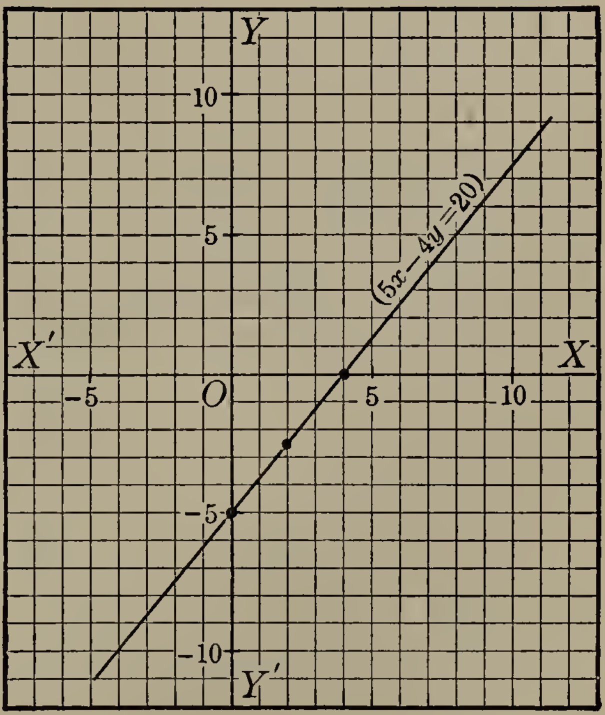
Find and tabulate six pairs of values for x and y which satisfy each of the following equations. Use numbers not greater than 10. Use at least one negative value for x and one negative value for y. Then plot the six corresponding points. Can a straight line be drawn through the six points obtained in each exercise?

2. $x - 5y = 6$.
3. $x - 2y = 4$.
4. $3x - 2y = 12$.
5. $2x - y = 0$.
6. $3x = 2y$.
7. $y = 2x - 1$.

110. Equation of a straight line. The preceding work should convince the student that the graph of an equation of the first degree in x and y is a *straight line.* This fact can be proved, but the proof is too difficult to be given at present. Therefore it will be assumed that the graph of every linear equation in two variables is a straight line. And since the position of a straight line is known when *any two* of its points are given, it will be sufficient in graphing a linear equation in two variables to plot *any two points* whose x- and y-distances satisfy the equation, and then to draw a straight line through these two points. The two points most convenient to plot are usually the two in which the line cuts the axes. Occasionally these points come very close together, and consequently they will not determine accurately the position of the line. In such cases one should decide on two values of x rather far apart (such as 0 and 5, or 0 and -5) and compute the corresponding values of y. Two such points will fix the position of the line with sufficient accuracy.

If a line goes through the origin (as in Exercise 5 preceding), $x = 0$, $y = 0$ will do for one point, but a point not on both axes must be taken for the second one.

The essentials of the method of graphing a given linear equation in x and y are illustrated in the following



EXAMPLE

Graph the equation

$$5\,x - 4\,y = 20.$$

Solution. In this equation, if $x = 0$, $y = -5$; and if $y = 0$, $x = 4$. Here the point $(0, -5)$ is on the y-axis in the adjacent figure, 5 units below the origin; and the point $(4, 0)$ is on the x-axis, 4 units to the right of the origin. The straight line through these two points is the graph of $5\,x - 4\,y = 20$.

The necessary work may be tabulated as follows:

$$5x - 4y = 20.$$

If $x =$	0	4	2
then $y =$	-5	0	$-2\frac{1}{2}$

Check. If an error has been made in obtaining the value of x or y from the equation, or in plotting the values found, it can be quickly detected by plotting a third point, as $(2, -2\frac{1}{2})$, the values of whose x- and y-distances satisfy the equation. If this third point lies on the line determined by the first two points, the line is probably correct; if it does not, an error has been made.

EXERCISES

Graph the following linear equations:

1. $x + y = 7$.
2. $x + y = 0$.
3. $5x - 2y = 9$.
4. $x - y = 0$.
5. $3y + 2x = 4$.
6. $4x + y = 3$.
7. $x - 2y = 0$.
8. $x + y = -5$.
9. $x = 5$.

HINT. The equation $x = 5$ is equivalent to the equation $x + 0y = 5$. The latter is satisfied by $x = 5$, and *any* value of y. Thus the pairs of values $(5, 1)$, $(5, 5)$, $(5, -2)$, etc. satisfy the equation $x + 0y = 5$. Plotting these points, it is evident that the required graph is a line parallel to the y-axis and 5 units to the right of it.

10. $y = -3$.
11. $x = 9$.
12. $x = 0$.
13. $x = -4$.
14. $y = 2$.
15. $y = 5$.

16. Is the point (4, 3) on the line whose equation is $2x - 3y = 12$? Is (0, 6)? Is (6, 0)?

17. If a point is on a line, do the values of its x-distance and its y-distance satisfy the equation of the line?

18. If the values of the x-distance and the y-distance of a point satisfy the equation of a line, is the point located on the graph of the equation?

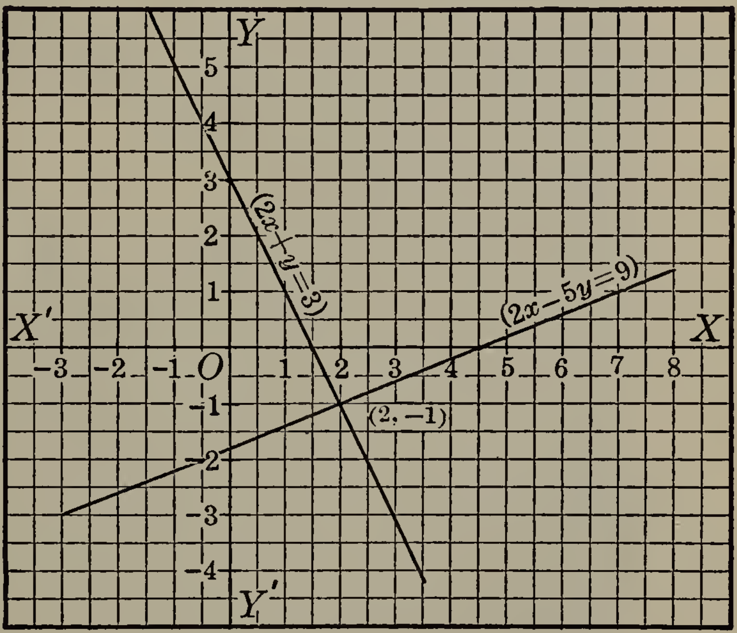
19. Determine without reference to the graph itself whether the point (6, 5) is on any of the graphs of the equations in Exercises 1–15 above. If so, on which does it lie?

It should now be clear that

The equation of a line is satisfied by the values of the x-distance and the y-distance of any point on that line.

Any point the values of whose x-distance and y-distance satisfy the equation is on the graph of the equation.

111. Graphical solution of linear equations in two variables. If we construct the graphs of the two equations $2x - 5y = 9$ and $2x + y = 3$ as indicated in the figure below, it is seen that for the point of intersection of the graphs x is 2 and y is -1. Since the point $(2, -1)$ is on both graphs, these values should satisfy both equations. Substituting 2 for x and -1 for y in each equation, we get the identities $4 - (-5) = 9$ and $4 + (-1) = 3$. Thus the *graphical solution* of two linear equations consists in plotting the two equations and finding from the graph the value of x and the value of y at the point of intersection. Since two straight lines *can intersect in but one point,* there can be but *one pair of values of x and y* which satisfies a pair of linear equations in two variables.



The necessary work is tabulated as follows:

$2x - 5y = 9$

If $x =$	0	$4\frac{1}{2}$	7
then $y =$	$-1\frac{4}{5}$	0	1

$2x + y = 3$

If $x =$	0	1	$1\frac{1}{2}$
then $y =$	3	1	0

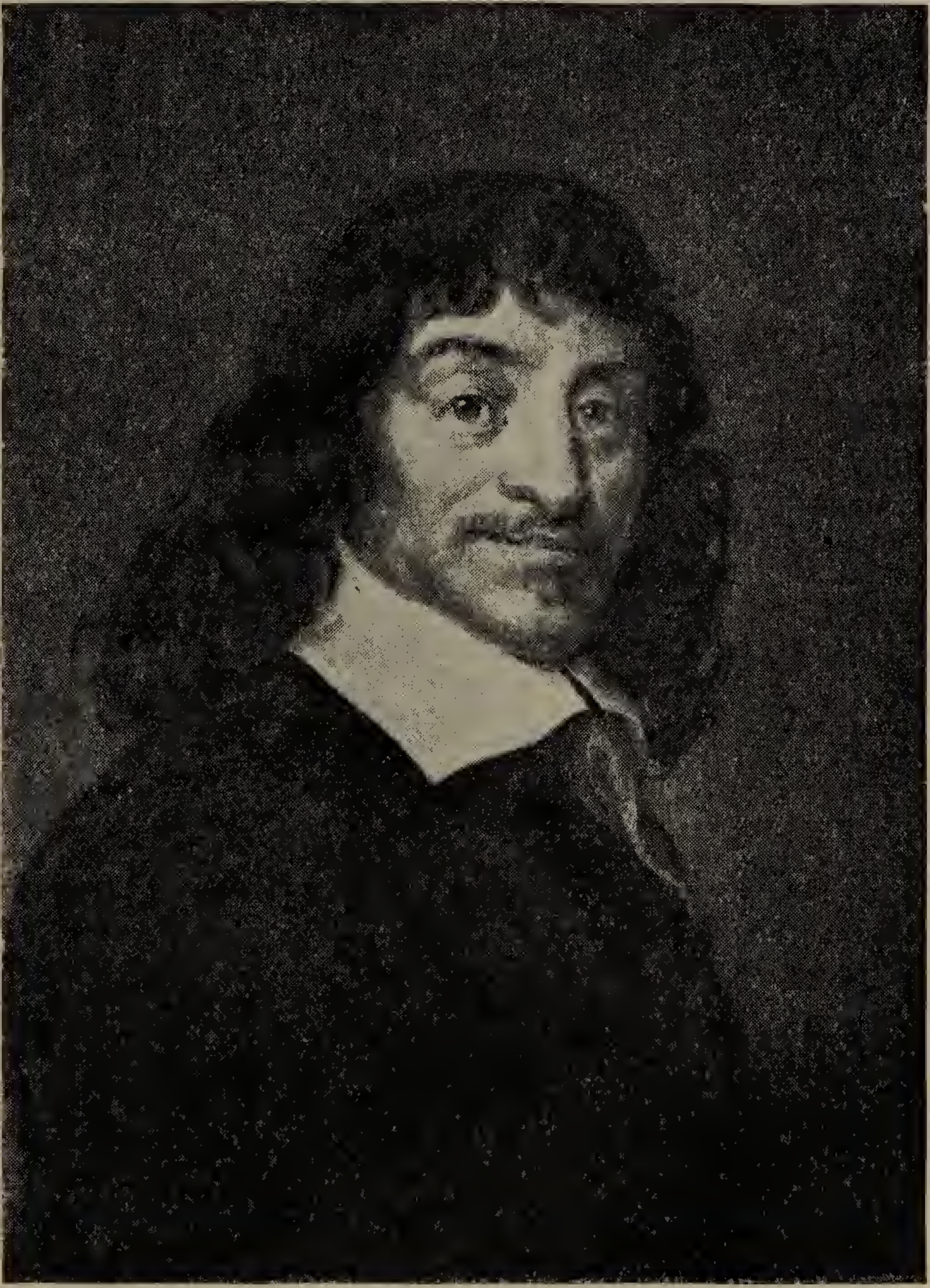
EXERCISES

Solve graphically the following systems and verify by substituting in each set of equations the values of x and y for the point of intersection as obtained from the graph:

1. $4x - 3y = -5,$
   $7x + y = 10.$
2. $x + y = 2,$
   $3x - 2y = 16.$
3. $2x + 3y = 13,$
   $x - y = 4.$
4. $4x + y = 17,$
   $x - y = -2.$
5. $14y + 12x = 13,$
   $2x + 6y = -7.$
6. $4x + 7y = 5,$
   $18y - 5x = -33.$
7. $x + 3y = 5,$
   $2x - 5y = -1.$
8. $x + 8y = 16,$
   $7y - 5x = 14.$
9. $2x + 2y = 3,$
   $4x - 6y = 1.$
10. $3x - 2y = 0,$
    $6x - 4y = 7.$
11. $x = 1,$
    $9x + 8y = 17.$
12. $x = 4y,$
    $2x - 3y = 5.$

BIOGRAPHICAL NOTE

René Descartes. One of the two or three most important advances ever made in mathematics was the discovery that algebraic equations could be represented geometrically. This great discovery was made by René Descartes (1596–1650), the French philosopher. Though never rugged in health, he took part in several campaigns when a young man, and it is said that during a weary winter spent in camp in Austria he first conceived the ideas that resulted in this important work. Though his writings read very differently from a modern book on the same subject, yet he developed all the essentials of graphical representation. He saw that a letter, that is, a coördinate, might represent either a positive or a negative number, and so forced upon mathematicians the conviction that negative integers are indeed numbers and that they are useful in algebraic operations. After his



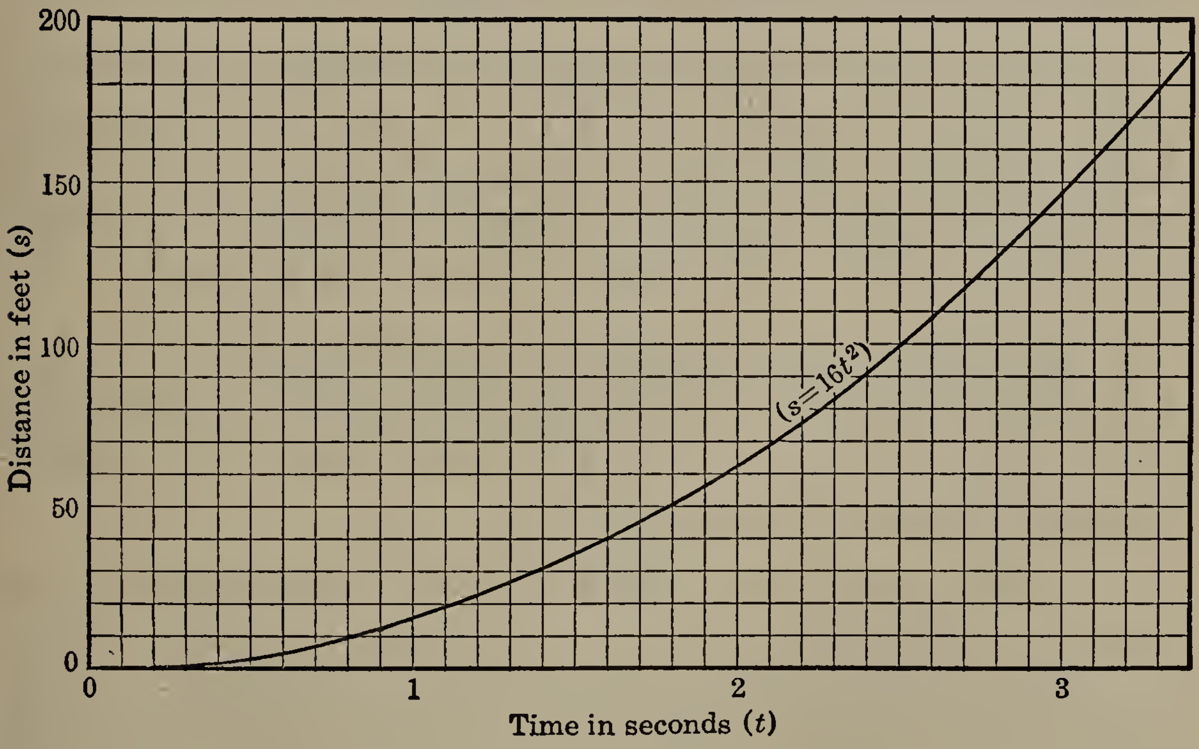
René Descartes

time they were not usually ruled out as absurd or impossible, as was commonly the case before. He also introduced the modern exponential notation. To Descartes is due the use of the last letters of the alphabet for the unknown and the first letters for the known numbers. Though the sign = was used long before his time, he did not accept it. He used the asterisk to indicate that a certain power of the variable was lacking. Thus he would have written the equation $x^3 - 8x + 16 = 40$ in the form $x^{3*} - 8x + 16 \infty 40$.

112. Graphs of formulas. In many cases it is possible to use a graph as a substitute for numerical computation.

Example

If a body falls from rest (neglecting the resistance of the air), the distance in feet, s, which it travels in t seconds is given by the formula $s = 16t^2$. Graph the formula.



From this graph estimate how far the body falls in $1\frac{1}{2}$ seconds; in $3\frac{1}{2}$ seconds. How long does it take for the body to fall 8 feet? 36 feet? 150 feet?

EXERCISES

1. One dollar at simple interest for t years at 6% will amount to a sum represented by the expression $A = 1 + .06\,t$. Draw the graph for this formula. From the graph tell how long it would take a dollar to double itself at this rate of interest. If a boy at the age of 10 had deposited a dollar in the bank at 6% interest, to how much would his account amount by the time he was 35 years of age?

2. A dollar at 6% interest compounded annually for a period of n years will increase in accordance with the formula $A = (1 + .06)^n$. Plot this formula. How long would it take a dollar to double under these conditions? If a boy 10 years old deposited one dollar in the bank at 6% interest compounded annually, how much would it amount to if he left it there from the time he was 10 years old until he was 35?

3. The circumference of a circle is expressed by the formula $C = 2\,\pi r$, where $\pi = 3.14$. Graph this equation. From the curve estimate the circumference of a circle which is 10 feet in diameter; 3 feet in diameter. A circle has a circumference of 12 feet; what is its radius?

4. The area of a circle is given by the expression $A = \pi r^2$. Graph this equation. If a circle has an area of 25 square feet, what is its radius? What is the area of a circular table-top $3\frac{1}{2}$ feet in diameter?

5. A falling body moves with a velocity represented by the formula $V = 32\,t$, where V is the velocity of the body in feet per second and t is the time from rest expressed in seconds. Graph this equation. From the graph find the time it will take before a body moves with a speed of 50 feet per second; 82 feet per second. How fast is the body moving after it has been falling for 5 seconds? 13 seconds? $\frac{1}{2}$ second?

6. The velocity of a jet of water flowing from a hole in a sheet-metal tank is expressed by the formula $V = 8\sqrt{H}$, where V is the velocity of the jet in feet per second and H is the height in feet (often called the "head") of the water level in the tank above the hole. Graph this formula for various heads up to 100 feet. By inspection of the curve tell what is the velocity when the head is 20 feet; 45 feet; 33 feet. What must the head be to give a velocity of 10 feet per second? $22\frac{1}{2}$ feet per second?

HINT. Substitute values 0, 1, 4, 9, 16, 25, 36, etc. for H in determining the corresponding values of V.

7. The horse power of an engine is expressed by the approximate formula $HP = \frac{nb^2}{2.5}$, where HP is the horse power, b the bore (or diameter) of one of the cylinders in inches, and n the number of cylinders. Graph this expression in the case of a six-cylinder engine. How many horse power would you get from a six-cylinder engine with a 1-inch cylinder? a 6-inch cylinder? What diameter of cylinder would you need to develop 100 HP? 25 HP?

8. The horse power required to drive an automobile over a smooth, level road against the resistance of the air is given by the following formula:

$$HP = \frac{0.0185V^3 + 147V}{550}.$$

In this equation HP is the required horse power and V is the speed of the car in miles per hour. Graph this formula and determine the following values: What would be the required power to drive the car 25 miles per hour? 100 miles per hour? What will be the maximum speed obtainable with a 50-HP motor? a 10-HP motor?

CHAPTER XX

SQUARE ROOT

113. Square root of algebraic expressions. Evidently

$$t^2 + 2tu + u^2 = \pm (t + u)^2.$$

A study of this form will enable us to extract the square root of any polynomial. Obviously, the square root of t^2 (the first term of the trinomial) is t, the first term of the root. If t^2 is subtracted from the trinomial, the remainder is $2tu + u^2$. The next term of the root (u) can be found by dividing the first term of the remainder ($2tu$) by $2t$ (twice that term of the root already found).

The work may be arranged thus:

$$\begin{array}{lll}
 & & \multicolumn{1}{l}{t+u} \\ \cline{3-3}
 & & \multicolumn{1}{l}{t^2 + 2tu + u^2} \\
 & & \multicolumn{1}{l}{t^2} \\ \cline{3-3}
\text{Trial divisor,} & 2t & \multicolumn{1}{|l}{2tu + u^2} \\
\text{Complete divisor,} & 2t + u & \multicolumn{1}{|l}{2tu + u^2 = (2t + u)u} \\ \cline{2-3}
\end{array}$$

Therefore the required roots are $\pm (t + u)$.

The foregoing process is easily extended to the polynomial $9x^4 - 24x^3 + 28x^2 - 16x + 4$, as follows:

$$\begin{array}{lll}
 & & \multicolumn{1}{l}{3x^2-4x+2} \\ \cline{3-3}
 & & \multicolumn{1}{l}{9x^4-24x^3+28x^2-16x+4} \\
 & (3x^2)^2= & \multicolumn{1}{l}{9x^4} \\ \cline{3-3}
\text{First trial divisor,} & 2\cdot 3x^2=6x^2 & \multicolumn{1}{|l}{-24x^3+28x^2} \\
\text{First complete divisor,} & 6x^2-4x & \multicolumn{1}{|l}{-24x^3+16x^2=(6x^2-4x)(-4x)} \\ \cline{3-3}
\text{Second trial divisor,} & & \multicolumn{1}{|l}{} \\
 & 2(3x^2-4x)=6x^2-8x & \multicolumn{1}{|l}{12x^2-16x+4} \\
\text{Second complete divisor,} & 6x^2-8x+2 & \multicolumn{1}{|l}{12x^2-16x+4=(6x^2-8x+2)2} \\ \cline{3-3}
\end{array}$$

Therefore the required roots are $\pm (3x^2 - 4x + 2)$.

The term $3x^2$ was obtained by taking the square root of $9x^4$; the second term, $-4x$, by dividing $-24x^3$ by the first trial divisor, $6x^2$; and the third term, 2, by dividing $12x^2$ by $6x^2$, the first term of the second trial divisor.

The method just illustrated of extracting the square root of a polynomial may be stated in the

Rule. *Arrange the terms of the polynomial according to descending or ascending powers of some letter in it.*

Extract the square root of the first term. Write the result (with plus sign only) as the first term of the root and subtract its square from the given polynomial.

Double the root already found for the first trial divisor, divide the first term of the remainder by it, and write the quotient as the second term of the root.

Annex the quotient just found to the trial divisor, making the complete divisor; multiply the complete divisor by the second term of the root and subtract the product from the last remainder.

If terms of the polynomial still remain, double the root already found for a trial divisor, divide the first term of the trial divisor into the first term of the remainder, write the quotient as the next term of the root, form the complete divisor, and proceed as before until the process ends, or until the required number of terms of the root have been found.

Inclose the root thus found in a parenthesis preceded by the sign ±.

Note. The process of extracting the square root of numbers was familiar to mathematicians long before they knew how to find the square root of polynomials. This is consistent with the fact that the development of the methods of performing operations on literal number symbols generally followed and grew out of the similar operations on numerals. The application of the rules for extracting the square root of numbers to that of polynomials is generally ascribed to Recorde (1510–1558), who was the author of the earliest English work on algebra that we know. This book, which bears the title "The Whetstone of Wit," gives an accurate idea of the algebraic knowledge of the time and had a very wide influence.

EXERCISES

Obtain the positive square root of:

1. $a^2 + 4a + 4$.
2. $16 + 8x + x^2$.
3. $9m^2 + 12mn + 4n^2$.
4. $4x^2 - 20xy + 25y^2$.
5. $25 + 20a + 4a^2$.
6. $36n^6 + 36n^4 + 9n^2$.
7. $x^8 + 2x^4y^2 + y^4$.
8. $4s^4 + 4s^2t + 4s^2r + 2tr + t^2 + r^2$.
9. $9c^2 - 12cb^3 + 4b^6$.
10. $4a^2 + 4ab - 16ac + b^2 - 8bc + 16c^2$.
11. $25a^4 + 20a^3b + 94a^2b^2 + 36ab^3 + 81b^4$.
12. $36n^4 + 60n^5 + 36n^2 + 25n^6 + 30n^3 + 9$.
13. $36x^6 - 24nx^3 + 12n^2x^3 + 4n^2 - 4n^3 + n^4$.
14. $4a^2 + 20ab + 8ab^2 + 25b^2 + 20b^3 + 4b^4$.
15. $c^2 + d^2 + x^4 - 2dx^2 + 2cd - 2cx^2$.
16. $100r^2 + 20rc^2 - 20rs + c^4 - 2c^2s + s^2$.
17. $\frac{9x^2}{4} + xy + \frac{y^2}{9}$.
18. $16a^4 + \frac{4a^2b^2}{9} + \frac{16a^3b}{3}$.
19. $36n^2 - 9mn + 12m^2n + \frac{9m^2}{16} - \frac{3m^3}{2} + m^4$.
20. $\frac{25x^2}{4} + \frac{4y^2}{49} - \frac{10xy}{7}$.
21. $\frac{36m^6}{25} + \frac{16m^3n^2}{5} + \frac{16n^4}{9}$.
22. $\frac{r^2}{4} + \frac{rs}{4} - \frac{rt}{3} + \frac{s^2}{16} - \frac{st}{6} + \frac{t^2}{9}$.

Find the first three terms in the approximate square root of:

23. $4 + 8M + 12M^2$.
26. $25x^2 + 10x + 4$.
24. $25x^2 + 40x + 36$.
27. $25t^2 + 18t + 16$.
25. $36 - 14a + 22a^2$.
28. $9x^2 + 16x + 4$.
29. $100x^4 - 100x^3 + 68x^2 - 20x + 4$.
30. $4x^4 + 17x^2 - 12x + 4 - 12x^3$.
31. $m^4 - 6m^3 + 10m^2 + \frac{1}{4} - 3m$.
32. $9r^4 - 12r^3 + 5r^2 + 10 - 2r$.

114. Square root of arithmetic numbers. Since $1 = 1^2$, and $81 = 9^2$, a one-digit or a two-digit square has only *one* digit in its square root.

And as $100 = 10^2$, and $9801 = (99)^2$, a three-digit or a four-digit square has *two* digits in its square root.

Also, $10{,}000 = 100^2$, and $998{,}001 = (999)^2$; hence a five-digit or a six-digit square has *three* digits in its square root.

These examples illustrate the relation between the number of digits in a number and the number of digits in its square root. They also suggest a method of obtaining the first digit in the square root of any number.

For example, take the four numbers 35 24 18, 3 52 41, .25 38, and .05 28 6. Beginning at the decimal point in each, point off groups of two digits each, as indicated. Any incomplete group of two digits on the right, as in .05 28 6, should be completed by annexing one zero; thus, .05 28 60. Now the first digit in the square root is the greatest integer whose square is less than or equal to the left-hand group. This is true whether the latter contains *two* digits or *one*. Hence the first digit in the square root of 35 24 18 is 5, in the square root of 3 52 41 is 1, in the square root of .25 38 is 5, and in the square root of .05 28 60 is 2.

Moreover, the number of digits in the square root of a perfect square is equal to the number of periods, provided a *single digit* remaining on the left is called a period.

115. Extraction of the square root of numbers. The square root of arithmetic numbers may be found by a procedure based on the method of extracting the square root of polynomials.

Just how t and u are involved in the square of $(t + u)$, or $t^2 + 2\,tu + u^2$, is obvious on inspection, because the parts t^2, $2\,tu$, and u^2 cannot be united into one term. In the square of an arithmetic number, however, the parts are united. Thus $(47)^2 = (40 + 7)^2 = 1600 + 560 + 49 = 2209$. Now it is clear how 40 and 7 are involved in $1600 + 560 + 49$, but it is not plain from 2209 alone. Pointing off, however, enables us to discover at once the first digit, 4, which is equivalent to 4 tens, or 40. With the exception of pointing off, the method of extracting the square root of an arithmetic number does not differ greatly from the method of extracting the square root of an algebraic expression. In fact, the formula which states that the square root of $t^2 + 2\,tu + u^2 = \pm (t + u)$ can be used to explain the two processes.

If t denotes the tens and u the units, $t^2 + 2\,tu + u^2$ is closely related to $1600 + 560 + 49$, t^2 being 1600, or $(40)^2$; u^2 being 49, or 7^2; and $2\,tu$ being $2 \cdot 40 \cdot 7$. Therefore the process of extracting the square root of 2209 may be based on these relations and the work arranged as shown on the following page.

$$
\begin{array}{r|l}
 & \underline{4\ \ 7,\text{ or }40+7} \\
 & \overset{\frown}{22}\ \overset{\frown}{09} \\
t^2=(40)^2 & 16\ 00 \\
\hline
2\,t=2\cdot 40=80 & \ 6\ 09 \\
2\,t+u=80+7 & \ 6\ 09=(80+7)7=(2t+u)u=2tu+u^2 \\
\hline
\end{array}
$$

Therefore ± 47 are the two square roots of 2209.

If the number has three digits in its square root, the work and explanations may be arranged thus:

$$
\begin{array}{r|l}
 & \underline{3\ \ 1\ \ 5,\text{ or }300+10+5} \\
 & \overset{\frown}{9}\ \overset{\frown}{92}\ \overset{\frown}{25} \\
t^2=(300)^2 & 9\ 00\ 00=30\text{ tens squared} \\
\hline
\text{First trial divisor,} & \\
2\,t=2\cdot 300=600 & 92\ 25 \\
\text{First complete divisor,} & \\
2\,t+u=600+10=610 & 61\ 00=(2\cdot 30\text{ tens}+10\text{ units})\times 10 \\
\hline
\text{Second trial divisor,} & \\
2\,t=2\cdot 310=620 & 31\ 25 \\
\text{Second complete divisor,} & \\
2\,t+u=620+5=625 & 31\ 25=(2\cdot 31\text{ tens}+5\text{ units})\times 5 \\
\hline
\end{array}
$$

Therefore ± 315 are the square roots of 99,225.

When the method and reasons for the process have become familiar, the work may be shortened by omitting the explanations and unnecessary zeros as follows:

$$
\begin{array}{r|l}
 & \underline{3\ \ 6} \\
 & \overset{\frown}{12}\ \overset{\frown}{96} \\
 & 9 \\
\hline
66 & 3\ 96 \\
 & 3\ 96 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & \underline{1\ \ 1\ \ 6} \\
 & \overset{\frown}{1}\ \overset{\frown}{34}\ \overset{\frown}{56} \\
 & 1 \\
\hline
21 & 34 \\
 & 21 \\
\hline
226 & 13\ 56 \\
 & 13\ 56 \\
\hline
\end{array}
$$

The foregoing method is the one commonly used for extracting the square root of a number. For it we have the

Rule. *Begin at the decimal point and point off as many groups of two digits each as possible: to the left if the number is an integer; to the right if it is a decimal; to both the left and the right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand group, and write this integer for the first digit of the root and directly over the group of digits used in determining it.

Square the first digit of the root, subtract its square from the first group, and annex the second group to the remainder.

Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root and directly over the group of digits used in determining it.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next group for a new dividend.

Double the part of the root already found for a new trial divisor and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal group in the number.

Check. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

Sometimes in using a trial divisor we obtain too great a quotient for the next digit of the root. This happens in obtaining the second digit of the square root of 32,301, where 2 into 22 gives 11. Obviously 10 and 11 are both impossible. If 9 is tried we get 9 · 29, or 261, which is greater than 223. Similarly, 8 is too great. But 7 · 27 = 189, which is less than 223. Therefore 7 is the second digit of the root.

$$\begin{array}{rl} & 1 \\ & \overline{\overset{\frown}{3}\overset{\frown}{23}\overset{\frown}{01}} \\ & 1 \\ 2 & \big|\overline{223} \end{array}$$

Occasionally the trial divisor gives a quotient less than 1. This indicates that the required root digit is 0, which should be written in the root and the work continued as usual. An instance of this kind occurs in finding the second digit in the square root of 942.49. The quotient of 4 ÷ 6 is $\frac{2}{3}$, which is not an integer. Therefore the second digit of the root is 0. Then the next period, 49, should be brought down. The new trial divisor will be 60, which will give 7 as the third digit of the root. The work can easily be completed, giving 30.7 as the square root.

$$\begin{array}{r|l}
 & \;\;3\;\;\;0 \\ \hline
 & \overset{\frown}{9}\,\overset{\frown}{42}.\overset{\frown}{49} \\
 & 9 \\ \hline
6 & 42
\end{array}$$

An attempt to extract the square root of 2 by annexing decimal periods of zeros and applying the rule becomes a never-ending process.

The number 2 has no exact square root, and no matter how far the work is carried, there is no final digit. As the work stands, we know that the square root of 2 lies between 1.414 and 1.415.

$$\begin{array}{r|l}
 & 1.4\;\;1\;\;4 \\ \hline
 & \overset{\frown}{2}.\overset{\frown}{00}\,\overset{\frown}{00}\,\overset{\frown}{00} \\
 & 1 \\ \hline
24 & 100 \\
 & \;\;96 \\ \hline
281 & 400 \\
 & 281 \\ \hline
2824 & 11900 \\
 & 11296 \\ \hline
\end{array}$$

116. Significant figures. The digits that are retained as *correct* are called the significant figures. The digit 0 may or may not be significant. In the first computation on this page it was significant because it indicated the correct result in that decimal place.

As a further example consider the number 3600. We may say that 3600 expresses the number 3629 correct to two digits, or to two significant figures. In this case neither of the two 0's is significant. The number 3629 correct to three places is 3630. The number 4692 expressed correct to three significant figures is $47\dot{0}0$, the first 0 being significant, the second 0 not significant.

ORAL EXERCISES

Express the following numbers correct to three significant figures:

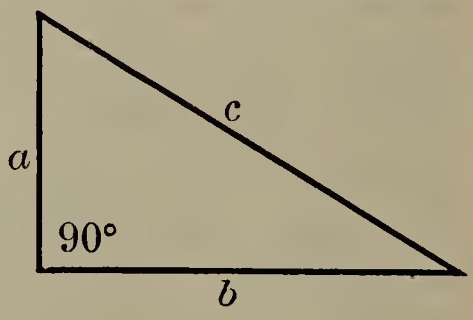
1. 27.61.
2. 3428.
3. 4969.
4. 9064.
5. 3.1416.
6. 3902.
7. 24.96.
8. 5,432,196.
9. 1.414.
10. 1.732.

EXERCISES

Find the square root of the following numbers, correct to the nearest hundredth:

1. 24.
2. 8.
3. 1.53.
4. 120.
5. 2.
6. 7.
7. 3.1416.
8. 549.
9. 10000.
10. 125.
11. 1563.
12. 2000.
13. 1.3.
14. .25.
15. 1924.
16. 5160.
17. 2395.
18. 8753.
19. 4,563,210.
20. 87,616.
21. 907,853.
22. 25,200.
23. 301,957.
24. 2,156,873.

Fact from Geometry. In the adjacent right triangle, $a^2 + b^2 = c^2$, the sides a and b, which form the right angle, are called the *sides*; and c, the side opposite the right angle, is called the *hypotenuse*.



If side a is 8 and side b is 15, then substituting in $a^2 + b^2 = c^2$ gives $64 + 225 = c^2$. Whence $289 = c^2$ and $c = \pm 17$.

Since -17 is not a practical answer, it is rejected.

In Exercises 25–28 find the hypotenuse and the area of a right triangle whose sides are:

25. 3 feet and 4 feet.
26. 16 inches and 12 inches.
27. 125 inches and 180 inches.
28. 293 inches and 873 inches.

In Exercises 29–32 find the hypotenuse and side of a right triangle whose area and other side are respectively:

29. 2 square feet and 1 foot.

30. 45 square inches and 15 inches.

31. 132 square feet and 12 feet.

32. 21 square inches and 3 inches.

In Exercises 33–35 find the other side of a right triangle in which the hypotenuse and one side are respectively:

33. 213 feet and 25 feet.
34. 8.5 miles and .29 mile.
35. 3.75 yards and .67 yard.

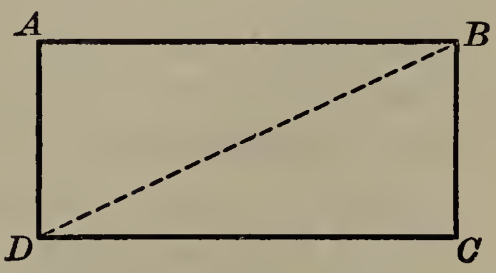
Extract the square root in Exercises 36–39 inclusive to the nearest ten thousandth, and in Exercises 40–44 inclusive to the nearest thousandth.

A fractional number should be reduced to a decimal before extracting the square root, unless the root is seen to be exact.

36. 3.14159.
37. 2.71828.
38. 1034.266.
39. .001432.
40. 5.
41. 7.
42. $\frac{11}{32}$.
43. $2\frac{7}{8}$.
44. $\frac{17}{64}$.

In the following find all roots to the nearest hundredth.

In rectangle *ABCD* line *DB* is called a *diagonal.*



45. Find the diagonal of a rectangle whose sides are 7 inches and 18 inches respectively.

46. The diagonal of a rectangle is 25 feet. Its perimeter is 70 feet. What are its length and its breadth?

47. One side of a rectangle is 10 inches. The diagonal is 15 inches. Find the perimeter and the area of the figure.

48. The side of a square is 25 inches. Find the diagonal.

49. The length of a rectangular figure is 23 feet. If the diagonal is twice the shorter side, what is the width?

50. The diagonal of a rectangle is 37 feet. The length is $1\frac{1}{2}$ times the width. Find the length and the breadth of the figure, and its area.

51. The diagonal of a rectangular figure is 25 feet. If the two sides are equal in length, what is the side? the area?

52. The distance traveled by a freely falling body is expressed by the formula $s = 16t^2$. In this expression s is the distance traveled in feet, while t is the time of fall in seconds. How long will it take a body to fall 100 feet? 40 feet?

53. The diagonal of a rectangle is twice the shorter side. The area of the figure is 580.378 square feet. Find the length, the breadth, and the diagonal of the figure.

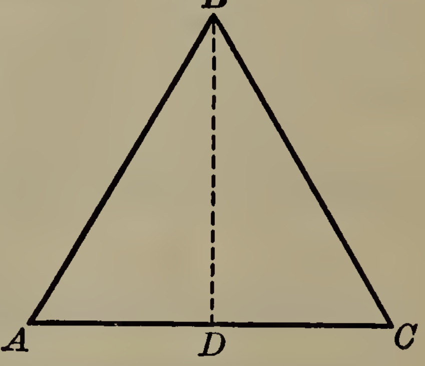
54. The width of a rectangle is 15 feet less than the length. The diagonal is 75 feet. Find the dimensions and the area.

55. The area of a circle is expressed by the equation $A = \pi r^2$. What is the radius (r) of a circle whose area (A) is 75 square feet? 123 square feet? ($\pi = 3.14$.)

Fact from Geometry. A line drawn from one vertex of an equilateral triangle to the middle point of the opposite side is perpendicular to that side.

Thus, in the equilateral triangle ABC, if D is the middle point of AC, BD is the altitude; and

$$\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2 = \overline{AB}^2 - \left(\frac{AC}{2}\right)^2.$$



56. The perimeter of an equilateral triangle is 18 inches. What is the area?

57. In the figure of the triangle given above, what is the ratio of the altitude to the side of the triangle?

58. Using the ratio obtained above, find the side of an equilateral triangle whose area is 118 square feet.

59. The formula for the amount (A) which a sum of money (P) amounts to at compound interest for a term of years (n) is $A = P(1+r)^n$ where r is the rate of interest expressed as a decimal fraction. What must be the interest rate to make $75 grow to $80 in 2 years? in 4 years?

60. A projectile fired horizontally from a gun will strike the level ground in a length of time the same as that for a body falling from rest the distance from the muzzle of the gun to the ground. The expression for this is $s = 16\,t^2$, where s is the distance in feet of the barrel of the gun above the level plain and t is the time in seconds. How far would a projectile travel if its average velocity were 1000 feet per second and the muzzle of the gun were 5 feet above the ground?

(The solution of the above problem neglects air resistance and the result obtained will therefore be only approximate.)

NOTE. A method of extracting the square root of numbers not unlike that in use today was employed by a Greek, Theon, about A. D. 350. In the Middle Ages square roots were extracted with a fair degree of accuracy by using the formulas of approximation:

$$(1)\quad \sqrt{a^2+x} = a + \frac{x}{2\,a}. \qquad\qquad (2)\quad \sqrt{a^2+x} = a + \frac{x}{2\,a+1}.$$

The true value of the square root of the number was proved to be between the results obtained by these expressions. Thus, if $\sqrt{65}$ was desired, it was noticed that $65 = 64 + 1$, and from (1)

$$\sqrt{65} = \sqrt{64+1} = \sqrt{8^2+1} = 8 + \frac{1}{2\cdot 8} = 8\tfrac{1}{16},$$

while from (2) $\sqrt{65} = \sqrt{64+1} = \sqrt{8^2+1} = 8 + \dfrac{1}{2\cdot 8+1} = 8\tfrac{1}{17}$.

Thus the true value of $\sqrt{65}$ is between these two numbers. This method was known to the Arabs.

It should be kept in mind that the use of decimal fractions and of the decimal point was not common until the eighteenth century. Consequently the complete development of the method of extracting the square root given in the text is comparatively recent.

REVIEW EXERCISES

Find the square root of the following numbers correct to four significant figures:

1. 3.1459.

2. 149,253.

3. 2,150,000.

4. 37,111,519.

5. 2,502,598.

6. 34,875,412.

7. 87.5476351.

8. 28,753.24935.

9. 3,587.281.

Factor the following expressions:

10. $10a^2 + ab - 3b^2$.

11. $10x^2 - x - 21$.

12. $12x^2 - 112x - 147$.

13. $1 + 24m^2 + 144m^4$.

14. $169 - x^2$.

15. $12x^2 - 11ax - 5a^2$.

16. The horse power transmitted by a leather belt in a shop is expressed by the formula $HP = \frac{(T-t)V}{33{,}000}$. T and t are the pulls of the belt on each side of the pulley, and V is the speed of the belt in feet per minute. What horse power is transmitted by a belt having pulls of 200 and 150 pounds respectively and a speed of 3000 feet per minute?

17. An automobile has a wheel 32 inches in diameter. How many revolutions per minute is the wheel making when the car is traveling 25 miles per hour? 30 miles per hour?

18. Two trains running between the same two stations take 20 minutes and 30 minutes respectively. One train travels 15 miles per hour faster than the other. What are their speeds?

19. A party contains 45 people. If there were 9 more women there would be half as many women as men. How many men and women are there in the party?

20. A man travels 25 miles, riding part of the way and walking the rest. If he walked 7 miles more than he rode, how far did he ride?

21. How much rope will be needed to reach from the top of a tent pole 10 feet high to a point on the ground 7 feet from the base of the pole?

CHAPTER XXI

RADICALS

117. Rational numbers. The quotient of two integers is called a *rational number.*

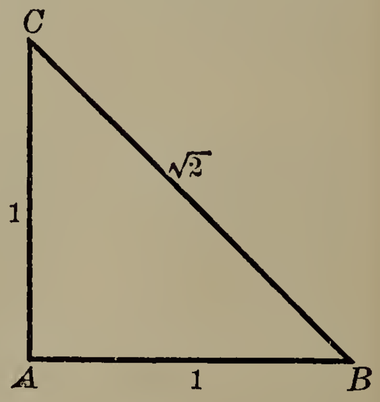
Any integer is a rational number, since it may be considered as the quotient of itself and 1.

Thus 4, -2, $1\frac{1}{3}$, $-\frac{8}{9}$, and 3.1416 are rational numbers. All fractions and decimal fractions are also rational numbers.

118. Radical. A *radical* is an indicated root of an algebraic or arithmetic expression.

Thus $\sqrt{19}$, $\sqrt{2}$, $\sqrt[4]{a}$, and $\sqrt{x^2 - 3x + 2}$ are radicals.

If a number under a radical sign is such that the root cannot be obtained exactly, the radical represents an *irrational number.*



Thus $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, are irrational numbers, since the indicated roots of 2, of 3, and of 5 will never come out even however far the process of extracting the root is carried.

Though no irrational numbers can be expressed exactly in decimals, we can represent some of them by the lengths of lines. Thus, in the right triangle ABC, if $AB = AC = 1$ inch, $BC = \sqrt{2}$ inches. If $AB = 2$ inches and $AC = 1$ inch, $BC = \sqrt{5}$ inches.

There are other types of irrational numbers which cannot be expressed in terms of radicals, but their consideration is too complicated for this text.

119. Imaginaries. If a number under a square-root sign is negative, the radical represents an *imaginary* number.

Thus $\sqrt{-2}$, $\sqrt{-8}$, and $\sqrt{-4}$ are imaginary numbers.

If the student pursues the study of algebra further he will learn that imaginary numbers are required to express completely the cube and higher roots of any positive or negative number.

For example, he will learn that the number 27 has *two other* cube roots besides the number 3. Imaginary numbers, however, are not considered in this text.

120. Index. The small figure, like the 3 in $\sqrt[3]{m}$, is called the *index* of the radical.

In $3\sqrt[4]{2}$, 4 is the index.

121. Radicand. The *radicand* is the number or expression under the radical sign.

In $\sqrt{3}$ and $\sqrt[3]{xy}$, 3 and xy are the radicands.

122. Principal root. If a number has two roots for a given index, the positive one is called the *principal root.*

When no plus or minus sign precedes the radical sign, the principal root is always implied.

For example, the number 4 has two square roots, $+2$ and -2. Here $+2$ is the principal square root of 4, and the radical $\sqrt{4}$ means $+2$ and not -2. Thus $\sqrt{4} = 2$, but $-\sqrt{4} = -2$.

In case a number has only one root for a given index, that root is the principal root.

For example, the principal root of $\sqrt[3]{-8}$ is -2; that of $\sqrt[3]{-27}$ is -3.

ORAL EXERCISES

In the following tell which number is the index and which is the radicand:

1. $\sqrt[3]{4}$. 2. $\sqrt[4]{16}$. 3. $\sqrt[5]{10}$. 4. $\sqrt{2}$. 5. $\sqrt[12]{13}$. 6. $\sqrt[10]{3}$.

Which of the following are rational numbers?

7. $\sqrt{4}$. 9. $\sqrt{12}$. 11. $\sqrt{3}$. 13. $\sqrt[3]{9}$. 15. $\sqrt[4]{1}$.

8. $\sqrt[3]{8}$. 10. $\sqrt[4]{81}$. 12. $\sqrt{2}$. 14. $\sqrt{8}$. 16. $\sqrt[3]{125}$.

What is the principal square root of the following?

17. 9. 18. 36. 19. 81.

What is the principal cube root of the following?

20. 8. 21. -27. 22. 125. 23. -125.

What is the principal fifth root of the following?

24. 1. 25. -1. 26. 32. 27. -32. 28. -243.

123. Simplification of radicals. The form of a radical expression may be changed without altering its numeric value. Such changes are necessary for many reasons. For example, the numeric value of a radical expression is most easily obtained from its simplest form. It will be made clear later that $\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$. Granting that the two fractions are really equal, one can see that the value to several decimal places can be computed more easily from the second fraction than from the first, since fewer long numeric operations are then necessary.

EXAMPLES

Study the following changes of form:

1. $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$
2. In general, $\sqrt{a^2b} = \sqrt{a^2}\sqrt{b} = a\sqrt{b}.$
3. Also, $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8}\sqrt[3]{3} = 2\sqrt[3]{3}.$
4. In general, $\sqrt[3]{a^3b} = \sqrt[3]{a^3}\sqrt[3]{b} = a\sqrt[3]{b}.$

The preceding examples 1 and 2 illustrate the following rule for simplifying a radical involving the square root of an integer or an integral expression.

RULE. *Separate the radicand into two factors, one of which is the greatest perfect square which the radicand contains. Then take the square root of this factor and write it as the coefficient of a radical of which the other factor is the radicand.*

If the radical already has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for radicals involving the cube and higher roots.

Thus $\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = 2\sqrt[3]{2},$

and $\sqrt[5]{96} = \sqrt[5]{32 \cdot 3} = 2\sqrt[5]{3}.$

NOTE. Although the Arabs were by no means able to state all the rules explained in this chapter, it is interesting to note that they did recognize the truth of a few of them. For instance, a writer about A.D. 830 gives, in his own notation of course, the facts contained in the formulas $\sqrt{a^2b} = a\sqrt{b}$ and $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

EXERCISES

Simplify:

1. $\sqrt{12}.$
2. $\sqrt{27}.$
3. $\sqrt{75}.$
4. $\sqrt{18}.$
5. $\sqrt{125}.$
6. $\sqrt{162}.$
7. $5\sqrt{20}.$
8. $\sqrt{32}.$
9. $\sqrt{48}.$
10. $3\sqrt{45}.$
11. $2\sqrt{98}.$
12. $5\sqrt{12}.$

13. $\sqrt{147}$.
14. $\sqrt[3]{16}$.
15. $\sqrt[3]{24}$.
16. $\sqrt[3]{54}$.
17. $2\sqrt[3]{128}$.
18. $3\sqrt[3]{40}$.
19. $5\sqrt[3]{96}$.
20. $\sqrt[3]{81}$.
21. $\sqrt{4\,x}$.
22. $\sqrt{3\,x^2}$.
23. $\sqrt{12\,m}$.
24. $\sqrt{5\,r^2}$.
25. $\sqrt{16\,a}$.
26. $\sqrt{4\,x^2}$.
27. $3\sqrt{9\,x}$.
28. $5\sqrt{5\,m^2}$.

29. $12\sqrt{8\,s^2}$.
30. $\sqrt{x^3}$.

HINT. $\sqrt{x^3} = \sqrt{x^2 \cdot x}$.

31. $7\sqrt{4\,x^3}$.
32. $\sqrt[3]{2\,x^3}$.
33. $\sqrt[3]{8\,x}$.
34. $\sqrt[3]{5\,m^3}$.
35. $\sqrt[3]{3\,t^3}$.
36. $\sqrt[3]{t^4}$.
37. $2\sqrt[3]{8\,m}$.
38. $4\sqrt[3]{2\,x^3}$.
39. $3\sqrt[3]{54\,x^3}$.
40. $12\sqrt{8\,x^4}$.
41. $\sqrt{(x+y)^2a}$.
42. $\sqrt{2\,x(b+c)^2}$.
43. $\sqrt{4(s+t)^2x}$.
44. $\sqrt{r(x-y)^2}$.
45. $3\sqrt{(x+z)^2\,4\,a}$.

46. $(a-b)\sqrt{(a+b)^2c}$.
47. $\sqrt{(a+b)(a^2-b^2)}$.
48. $\sqrt{(x-y)(x^2-y^2)}$.
49. $\sqrt{(r+2)(r^2-4)}$.
50. $\sqrt{x^2+4\,x+4}$.
51. $\sqrt{2\,x^2-8\,x+8}$.
52. $\sqrt{3\,x^2-30\,x+75}$.
53. $\sqrt{5\,r^2+5-10\,r}$.

124. Addition and subtraction of radicals. Radicals having the same index and identical radicands are really similar terms. They can be added or subtracted and the result expressed as one term according to the rule on page 48.

Thus $$2\sqrt{3} - 4\sqrt{3} + 5\sqrt{3} = 3\sqrt{3}.$$

If the radicands are not identical and cannot be simplified further, the radicals are really dissimilar terms, and addition or subtraction can then only be indicated (see page 50).

Thus $\sqrt{5}$, $\sqrt{3}$, and $\sqrt[3]{6}$ are three dissimilar radicals, and the addition of the three can only be indicated thus: $\sqrt{5}+\sqrt{3}+\sqrt[3]{6}$.

ORAL EXERCISES

Simplify:

1. $2\sqrt{2}+\sqrt{2}$.
2. $5\sqrt{3}-2\sqrt{3}$.
3. $4\sqrt{2}+3\sqrt{2}$.
4. $12\sqrt{5}-\sqrt{5}$.
5. $3\sqrt{6}-2\sqrt{6}$.
6. $4\sqrt{3}+5\sqrt{3}-6\sqrt{3}$.
7. $4\sqrt{7}-2\sqrt{7}+3\sqrt{7}$.
8. $7\sqrt{3}+4\sqrt{12}$.
9. $\sqrt{20}+2\sqrt{5}$.
10. $3\sqrt{8}+3\sqrt{2}-5\sqrt{32}$.
11. $6\sqrt{5}+10\sqrt{20}-3\sqrt{125}$.
12. $3\sqrt{x}+2\sqrt{x}$.
13. $2\sqrt{x}+5\sqrt{4x}$.
14. $3\sqrt{16a}+5\sqrt{4a}$.
15. $\sqrt{b}+\sqrt{a^2b}$.
16. $a\sqrt{75}-\sqrt{48a^2}$.

EXERCISES

Simplify and collect:

1. $\sqrt{8}+\sqrt{2}+2\sqrt{32}$.
2. $\sqrt{12}+\sqrt{27}$.
3. $\sqrt{3}+\sqrt{48}$.
4. $2\sqrt{32}-5\sqrt{18}$.
5. $3\sqrt{40}+\sqrt{250}$.
6. $4\sqrt{27}+2\sqrt{147}$.
7. $\sqrt{8}+\sqrt{32}-\sqrt{50}$.
8. $15\sqrt{75}-\sqrt{27}+6\sqrt{12}$.
9. $3\sqrt{27}-5\sqrt{3}$.
10. $\sqrt{5}+3\sqrt{125}$.
11. $3\sqrt{x^3}+5x\sqrt{x}$.
12. $\sqrt{a^3}+a\sqrt{a}+\frac{1}{a}\sqrt{a^5}$.
13. $\sqrt{a^2b}+\sqrt{a^2b^3}+\sqrt{b}$.
14. $3\sqrt{x}+4\sqrt{9x}+5\sqrt{4x}$.
15. $a\sqrt{\frac{x}{a^2}}+3\sqrt{x}+2\sqrt{16x}$.
16. $x+3\sqrt{3x^2}$.
17. $\sqrt{a^3}+\sqrt{b^2a}+\sqrt{4a}$.
18. $2\sqrt{x}-3\sqrt{4x}+\frac{1}{x}\sqrt{144x^3}$.
19. $\sqrt{a^3b^2c}-a\sqrt{ab^2c^3}+a\sqrt{ac}$.
20. $4\sqrt{3a^3}+\sqrt{12a}-2\sqrt{27a^5}$.

In the following exercises find the value of the expression, first by extracting the square root of each term and adding the results, then by simplifying and collecting before extracting the square root. Compare the amount of labor involved in the two methods.

21. $3\sqrt{50}+2\sqrt{2}-12\sqrt{98}$. 22. $4\sqrt{27}-5\sqrt{75}+2\sqrt{3}$.

23. $\sqrt{125}+2\sqrt{80}-3\sqrt{20}+5\sqrt{45}$.

125. Simplification of fractional radicands. Radicals whose radicands are fractions or fractional expressions occur frequently in practice, especially in certain parts of geometry.

EXAMPLES

Study the following simplification of fractional radicands:

1. $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\sqrt{\frac{1}{9}\cdot 6}=\sqrt{\frac{1}{9}}\sqrt{6}=\frac{1}{3}\sqrt{6}$.
2. $6\sqrt{\frac{2}{5}}=6\sqrt{\frac{10}{25}}=6\sqrt{\frac{1}{25}\cdot 10}=6\cdot\frac{1}{5}\sqrt{10}=\frac{6}{5}\sqrt{10}$.
3. $\sqrt{\frac{1}{2x}}=\sqrt{\frac{2x}{4x^2}}=\sqrt{\frac{1}{4x^2}\cdot 2x}=\frac{1}{2x}\sqrt{2x}$.
4. $\sqrt{\frac{a}{b}}=\sqrt{\frac{ab}{b^2}}=\sqrt{\frac{1}{b^2}\cdot ab}=\frac{\sqrt{ab}}{b}$.

These examples illustrate the following rule for simplifying an indicated square root which has a *fractional* radicand.

RULE. *Multiply the numerator and the denominator of the radicand by the least whole number or the simplest expression which will make the resulting denominator a perfect square.*

Then separate the radicand into two factors, one of which is a fraction and at the same time the greatest perfect square which the radicand contains.

Take the square root of this factor and write it as the coefficient of the radical of which the other factor is the radicand. If the original radical has a coefficient, multiply the result as obtained above by this coefficient.

ORAL EXERCISES

Simplify:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{2}{7}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{\frac{1}{a}}$.
5. $3\sqrt{\frac{1}{3}}$.
6. $4\sqrt{\frac{3}{4}}$.
7. $\sqrt{\frac{x}{y}}$.
8. $\sqrt{\frac{2x}{a}}$.
9. $a\sqrt{\frac{b^2}{a}}$.
10. $\sqrt{\frac{3}{5}}$.

EXERCISES

Simplify the following:

1. $\sqrt{\frac{5}{16}}$.
2. $\sqrt{\frac{3}{2}}$.
3. $\sqrt{\frac{2}{9}}$.
4. $\sqrt{\frac{7}{8}}$.
5. $\sqrt{\frac{4}{5}}$.
6. $\sqrt{\frac{5}{3}}$.
7. $\sqrt{\frac{3}{8}}$.
8. $\sqrt{\frac{13}{21}}$.
9. $2\sqrt{\frac{15}{16}}$.
10. $3\sqrt{\frac{2}{3}}$.
11. $5\sqrt{\frac{7}{2}}$.
12. $2\sqrt{\frac{16}{3}}$.
13. $5\sqrt{\frac{9}{5}}$.
14. $3\sqrt{\frac{8}{15}}$.
15. $7\sqrt{\frac{5}{8}}$.
16. $\sqrt{\frac{6m^3}{10}}$.
17. $m\sqrt{\frac{21b}{m}}$.
18. $5\sqrt{\frac{b^2a}{8}}$.
19. $\sqrt{\frac{4}{5}}+\sqrt{\frac{1}{5}}$.
20. $3\sqrt{\frac{5}{3}}-2\sqrt{60}$.
21. $4\sqrt{\frac{1}{3}}+2\sqrt{\frac{75}{4}}$.
22. $\sqrt{4}-3\sqrt[3]{8}+5\sqrt{9}$.
23. $2\sqrt[3]{27}+5\sqrt{\frac{25}{4}}-3\sqrt{225}$.
24. $4\sqrt{2}+13\sqrt{8}-5\sqrt{162}$.
25. $15\sqrt[3]{16}+5\sqrt[3]{54}-3\sqrt[3]{250}$.
26. $\sqrt{16}+\sqrt[3]{64}+\sqrt[4]{256}$.
27. $\dfrac{3\sqrt{3}+4\sqrt{27}+15\sqrt{75}}{\sqrt{48}}$.
28. $3\sqrt{8}+5\sqrt{\frac{1}{2}}-6\sqrt{2}+7\sqrt{\frac{64}{8}}$.
29. $5\sqrt{12}+2\sqrt{27}-4\sqrt{3}+3\sqrt{\frac{1}{3}}$.
30. $\sqrt{\frac{16}{7}}+3\sqrt{28}-5\sqrt{\frac{4}{7}}$.
31. $3\sqrt{4\frac{1}{6}}+2\sqrt{24}-9\sqrt{\frac{2}{3}}$.
32. $2\sqrt{5}-3\sqrt{\frac{1}{5}}+6\sqrt{125}-6\sqrt{\frac{36}{5}}$.
33. $3\sqrt{2}+5\sqrt{4}-2\frac{1}{2}\sqrt{18}+8\sqrt{\frac{1}{8}}$.

34. $5\sqrt{x^2y + 2\,xy + y} - 2\sqrt{(x+1)^2y}$.

35. $3\sqrt{4\,x^3 + 4\,x^2 + x} + 4\sqrt{(x+1)^2x}$.

36. $12\sqrt{ax^2 - 2\,ax + a} - 3\sqrt{(x-1)^2a^3}$.

The need for simplifying radicals presents itself in various problems, as, for example, in geometric work on right triangles.

PROBLEMS

(Obtain answers in *simplest radical form.*)

1. One side of a right triangle is 6 and the other side is 9. Find the hypotenuse.

Solution. $h = \sqrt{36 + 81} = \sqrt{117} = 3\sqrt{13}$.

2. The hypotenuse of a right triangle is 15 and one side is 10. What is the other side? What is the area?

3. The side of a square is 3. What is the diagonal?

4. The diagonal of a rectangle is D, and the width is $\frac{D}{3}$. Find the length and the area.

5. Find the area of a rectangle, of diagonal D, if the length is twice the breadth.

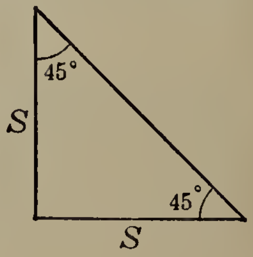
6. Find the sides of a square if the diagonal is 10.

7. What is the side of a square whose diagonal is $3\,L$?

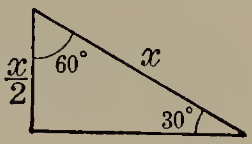
Problems involving the following classes of triangles are of frequent occurrence in practical work and often require the use of radicals:

(*a*) An isosceles right triangle; that is, a right triangle with two equal sides.

As indicated in the figure, if each side is S the two acute angles are 45° each.

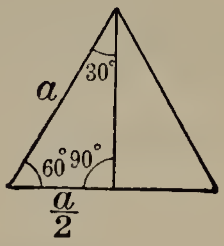


(*b*) A right triangle with one angle 30° or 60°.



As indicated in the figure, if one acute angle is 30° the other is 60°, and vice versa. More important still, the *hypotenuse* is always *twice the shorter side.*

(*c*) An equilateral triangle.



As indicated in the figure, when the altitude is drawn, it divides the base into two equal parts, and in each of the right triangles formed the same relations exist as in (*b*) above.

8. If each of the equal sides of an isosceles right triangle is 4, what is the hypotenuse?

HINT.
$$h^2 = 16 + 16.$$
$$h = \sqrt{16 + 16}, \text{ etc.}$$

9. The hypotenuse of an isosceles right triangle is 15. What is one side?

10. The formula for the area of a circle in terms of its radius is $A = \pi r^2$. What is the radius in terms of the area?

11. What is the radius of a circle whose area is 24 square inches?

12. The formula $C = \frac{5}{9}\,(F - 32)$ expresses the temperature by the centigrade scale in terms of the readings on the Fahrenheit scale. Solve this equation for F. What is the centigrade temperature corresponding to a Fahrenheit reading of 100°? 20°?

13. The area of the surface of a sphere in terms of the radius is $A = 4\pi r^2$. Find the radius in terms of the area.

14. What is the radius of a sphere whose surface is 18 square feet?

15. The volume of a cylinder in terms of its radius and height is $V = \pi r^2 h$. What is the height in terms of V and r? Find r in terms of h and V.

16. What must be the radius of a cylinder whose volume is 50 cubic inches and whose height is 20 inches?

17. A runway has an inclination of 30° to the horizontal. How far above the ground is a man who has walked along the runway a distance of 40 feet with respect to the horizontal?

HINT. See diagram in (*b*) on page 311.

18. A ladder makes an angle of 60° with the ground. A man climbs 15 feet up along the ladder. How far is he above the ground?

19. A man desires to make a bank of earth against a perpendicular wall 10 feet high. If the angle of repose of the earth is 45°, how many cubic feet of earth will he need to make a bank 20 feet long? How many square feet of sod will be required to cover the face of the bank?

HINT. The volume of earth required is equal to the length of the bank times the area of cross section, which is in this case an isosceles right triangle of 10-foot side.

20. What is the hypotenuse of an isosceles right triangle whose area is 50 square feet? 25 square feet?

21. The side of an equilateral triangle is 3 feet. Find the altitude and area.

22. The area of an equilateral triangle is $100\sqrt{3}$ square inches. What is the length of a side?

23. A hardware dealer desires to pack triangular files in a box. The files are $\frac{4}{10}$ of an inch on a side. What should be the depth of a box which will just contain one layer of files?

24. The hypotenuse of a right triangle with a 30° angle is 10 feet. What is the area?

NOTE. Though methods of classifying irrational expressions are found in the work of Euclid, the Hindus and the Arabs were the first to develop this part of algebra in a form similar to that used today.

126. Multiplication of radicals. Monomial radicals having the same index are multiplied as shown in the following:

EXAMPLES

1. Multiply $4\sqrt{2}$ by $3\sqrt{6}$.

Solution. $4\sqrt{2} \cdot 3\sqrt{6} = 12\sqrt{12} = 12 \cdot 2\sqrt{3} = 24\sqrt{3}$.

2. Multiply $3\sqrt[3]{2\,mr^2}$ by $2\sqrt[3]{5\,m^2r^2}$.

Solution. $3\sqrt[3]{2\,mr^2} \cdot 2\sqrt[3]{5\,m^2r^2} = 6\sqrt[3]{10\,m^3r^4} = 6\,mr\sqrt[3]{10\,r}$.

The method just illustrated of multiplying monomial radicals of the same index is stated in the

RULE. *Write the product of the coefficients of the radicals for the coefficient of the radical in the result.*

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for radicals having different indices.

ORAL EXERCISES

Perform the indicated multiplication:

1. $\sqrt{2} \cdot \sqrt{3}$.
2. $\sqrt{2} \cdot \sqrt{2}$.
3. $\sqrt{2} \cdot \sqrt{5}$.
4. $\sqrt{3} \cdot \sqrt{6}$.
5. $3\sqrt{2} \cdot 2\sqrt{3}$.
6. $5\sqrt{3} \cdot 2\sqrt{7}$.
7. $3\sqrt{2} \cdot 2\sqrt{5} \cdot 2\sqrt{3}$.
8. $5\sqrt{3} \cdot 4\sqrt{5}$.
9. $4\sqrt{13} \cdot 2\sqrt{2}$.

EXERCISES

Perform the indicated operation and simplify results?

1. $\sqrt{5} \cdot \sqrt{6}$.
2. $\sqrt{8} \cdot \sqrt{3}$.
3. $2\sqrt{15} \cdot 5\sqrt{12}$.
4. $3\sqrt{3} \cdot 2\sqrt{30}$.
5. $4\sqrt[3]{16} \cdot 10\sqrt[3]{2}$.
6. $6\sqrt{21} \cdot 5\sqrt{7}$.
7. $\sqrt{8} \cdot 4\sqrt{7}$.
8. $5\sqrt{18} \cdot \sqrt{3}$.
9. $m\sqrt{10} \cdot n\sqrt{5}$.
10. $\sqrt{x} \cdot b^2\sqrt{y}$.
11. $\sqrt{\frac{25\,a}{b}} \cdot \sqrt{\frac{4\,a^3}{b}}$.
12. $\sqrt{x^3} \cdot 5\sqrt{\frac{3}{x}}$.
13. $10\sqrt{m} \cdot 5\sqrt{\frac{1}{m}}$.
14. $5\sqrt{x} \cdot 5\sqrt{x}$.
15. $\sqrt{\frac{2\,m}{b}} \cdot \sqrt{\frac{2\,b}{m}}$.
16. $\sqrt{\frac{6\,x^3}{n}} \cdot \sqrt{\frac{n^3}{5\,x}}$.
17. $\sqrt{\frac{2\,m}{3}} \cdot \sqrt{\frac{3\,m}{2}}$.
18. $12\sqrt{\frac{x^3}{5}} \cdot 4\sqrt{\frac{x}{4}}$.
19. $2\sqrt{\frac{x^2}{y}} \cdot x\sqrt{\frac{y^3}{2}}$.
20. $\sqrt{2}(\sqrt{2} + 1)$.

HINT. $\sqrt{2}(\sqrt{2} + 1) = \sqrt{4} + \sqrt{2}$, etc.

21. $\sqrt{3}(\sqrt{3} - \sqrt{2})$.
22. $\sqrt{2}(2\sqrt{2} - \sqrt{5})$.
23. $\sqrt{5}(\sqrt{3} - 2\sqrt{2})$.
24. $2\sqrt{3}(\sqrt{3} - \sqrt{2})$.
25. $3\sqrt{5}(\sqrt{2} - \sqrt{3} + \sqrt{5})$.
26. $(\sqrt{5} - \sqrt{8} - \sqrt{3})(\sqrt{3})$.
27. $(\sqrt{2} - \sqrt{10} + \sqrt{20})(-2\sqrt{5})$.

Find the radical expression having the coefficient 1 which is equivalent to each of the following:

28. $4\sqrt{3}$.

Solution. $4\sqrt{3} = \sqrt{16}\sqrt{3}$
$= \sqrt{48}$.

29. $5\sqrt{5}$.
30. $3\sqrt{2}$.
31. $10\sqrt{13}$.
32. $4\sqrt{\frac{1}{2}}$.
33. $3\sqrt{\frac{1}{3}}$.
34. $5\sqrt{\frac{3}{5}}$.

35. $a\sqrt{x}$.

36. $3b\sqrt{y}$.

37. $4a\sqrt{2z}$.

38. $2x\sqrt{\frac{a}{y}}$.

39. $(x+y)\sqrt{\frac{3}{x+y}}$.

The multiplication of binomial or of polynomial radical expressions of the same order involves no new principle.

EXAMPLE

Multiply $3\sqrt{5}-4\sqrt{3}$ by $2\sqrt{5}+\sqrt{3}$.

Solution.

$$\begin{array}{llll} 3\sqrt{5} & -4\sqrt{3} & & \\ 2\sqrt{5}+ & \sqrt{3} & & \\ \cline{1-2} 30 & -8\sqrt{15} & & \\ & +3\sqrt{15} & -12 & \\ \cline{1-3} 30 & -5\sqrt{15} & -12 & =18-5\sqrt{15}. \end{array}$$

EXERCISES

Perform the indicated multiplication and simplify the results:

1. $(\sqrt{2}+5)(3\sqrt{2}-4)$.
2. $(7\sqrt{3}+2)(\sqrt{3}-5)$.
3. $(6\sqrt{5}-2)(\sqrt{5}+3)$.
4. $(4+3\sqrt{5})^2$.
5. $(5\sqrt{5}+2\sqrt{3})(2\sqrt{5}-3\sqrt{3})$.
6. $(2\sqrt{3}+5\sqrt{2})(2\sqrt{3}+2\sqrt{2})$.
7. $(3\sqrt{6}-2\sqrt{5})(12\sqrt{5}+13\sqrt{6})$.
8. $(4+\sqrt{2}-2\sqrt{3})(\sqrt{5}+\sqrt{2}+6)$.
9. $(3-2\sqrt{3}+\sqrt{2})(3\sqrt{2}+\frac{1}{2}\sqrt{\frac{1}{3}})$.
10. $(\sqrt{10}+\sqrt{15}-\sqrt{18})(\sqrt{2}+\sqrt{60}+3\sqrt{90})$.
11. $(\sqrt{x}+3\sqrt{y})(\sqrt{x}-3\sqrt{y})$.
12. $(4\sqrt{m}+3\sqrt{n})(3\sqrt{m}+4\sqrt{n})$.

13. $(2\sqrt{x}+3\sqrt{y})(\sqrt{x}-2\sqrt{y})$.

14. $(2\sqrt{3\,x}-5\sqrt{2\,y})(3\sqrt{12\,x}+2\sqrt{2\,y})$.

15. $\sqrt{x^2-y^2}\cdot\sqrt{3\,(x+y)}$.

16. $\sqrt{(a^2+2\,ab+b^2)^3}\cdot\sqrt{(a^2-b^2)(a+b)}$.

17. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(3+\sqrt{2})$.

18. $(3\sqrt{6}+2\sqrt{7}+3\sqrt{14})(4-\sqrt{7})$.

19. $(3\sqrt{x+1}-4\sqrt{x-1})(\sqrt{x^2-1})$.

20. $(2\sqrt{x-a}+3\sqrt{x}-4\sqrt{a})(\sqrt{x-a})$.

21. $(\sqrt{m+t}-2\sqrt{m}+3\sqrt{m-t})(\sqrt{m})$.

22. $(3\sqrt{2+x}-5\sqrt{x-2}+13\sqrt{x})(4\sqrt{x+2})$.

23. The hypotenuse of a right triangle is R, and one side is $\frac{R\sqrt{3}}{2}$. Find the other side.

24. One side of a right triangle is $\sqrt{2\,(R\sqrt{2}-1)}$, and the hypotenuse is R. What is the other side?

25. The sides of a right triangle are $\frac{1}{2}\,x\sqrt{2\sqrt{2}-2}$ and $\frac{1}{2}\,x\sqrt{2\sqrt{2}+2}$. What is the hypotenuse of the triangle? the area?

26. Show by substituting and simplifying that $2+2\sqrt{3}$ is a root of the quadratic equation $x^2-4\,x=8$.

27. Does $\frac{1}{2}(5-\sqrt{5})$ satisfy the equation $x^2+5=5\,x$?

127. Rationalization and division by radicals. It is frequently necessary to find the approximate value of an expression which involves division by a radical expression. Thus $3\div\sqrt{6}$, $\sqrt{2}\div\sqrt{7}$, are types which often occur.

To find the approximate value of $3\div\sqrt{6}$ we may extract the square root of 6 to several decimal places and then divide 3 by the approximate root obtained. Both these processes are long and one of them is unnecessary. For, writing $3 \div \sqrt{6}$ as a fraction and multiplying both terms by $\sqrt{6}$ gives $\frac{3\sqrt{6}}{6}$. To find the approximate value of this last fraction requires but one long operation.

One radical expression is the *rationalizing factor* for another if the *product* of the two is a *rational number.*

A rationalizing factor of $\sqrt{2}$ is $\sqrt{2}$, since $\sqrt{2} \cdot \sqrt{2} = 2$. Similarly, $\sqrt{3}$ is a rationalizing factor of $\sqrt{27}$, since

$$\sqrt{3} \cdot \sqrt{27} = \sqrt{81} = 9.$$

ORAL EXERCISES

Determine a rationalizing factor for each of the following expressions and find the product of the given expression and this factor:

1. $\sqrt{2}$.
2. $2\sqrt{5}$.
3. $\sqrt{3}$.
4. $\sqrt{10}$.
5. $3\sqrt{7}$.
6. $\sqrt{48}$.
7. $5\sqrt{24}$.
8. $\sqrt{8}$.
9. $\sqrt{14}$.
10. $2\sqrt{x}$.

Direct division of similar radicals, coefficient by coefficient and radicand by radicand, is often possible.

Thus $$\sqrt{6} \div \sqrt{2} = \sqrt{3}.$$

And $$12\sqrt{10} \div 2\sqrt{5} = 6\sqrt{2}.$$

$$(\sqrt{10} - \sqrt{15}) \div \sqrt{5} = \sqrt{2} - \sqrt{3}.$$

But $$\sqrt{7} \div \sqrt{3} = \frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{7} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{21}}{3}.$$

If direct division cannot be exactly performed, we use the

Rule. *Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the fraction by the rationalizing factor of the denominator and simplify the resulting fraction.*

EXERCISES

Rationalize the denominator of the following expressions:

1. $\dfrac{\sqrt{10}}{\sqrt{5}}$.

2. $\dfrac{\sqrt{6}}{\sqrt{2}}$.

3. $\dfrac{\sqrt{9}}{\sqrt{3}}$.

4. $\dfrac{\sqrt{3}}{\sqrt{2}}$.

5. $\dfrac{3\sqrt{7}}{2\sqrt{3}}$.

6. $\dfrac{\sqrt{6}}{\sqrt{12}}$.

7. $\dfrac{15\sqrt{24}}{5\sqrt{12}}$.

8. $\dfrac{\sqrt{21}}{5\sqrt{7}}$.

9. $\dfrac{\sqrt{5}-\sqrt{10}}{\sqrt{2}}$.

10. $\dfrac{3\sqrt{2}+5\sqrt{3}}{2\sqrt{3}}$.

11. $\dfrac{5\sqrt{15}}{4\sqrt{12}}$.

12. $\dfrac{3}{\sqrt{b}}$.

13. $\dfrac{5}{2\sqrt{x}}$.

14. $\dfrac{3y}{21\sqrt{a}}$.

15. $\dfrac{\sqrt{a}}{\sqrt{b}}$.

16. $\dfrac{2\sqrt{x}}{3\sqrt{2y}}$.

17. $\dfrac{a\sqrt{s}}{c\sqrt{t}}$.

18. $\dfrac{12\sqrt{x}}{5\sqrt{3y}}$.

19. $\dfrac{4\sqrt{54}}{3\sqrt{50}}$.

20. $\dfrac{2a\sqrt{x}}{3\sqrt{2x^3}}$.

21. $\dfrac{4\sqrt{mn}}{3m\sqrt{a}}$.

22. $\dfrac{12\sqrt{x}}{13a\sqrt{y}}$.

23. $\dfrac{13\sqrt{50}}{10\sqrt{75}}$.

24. $\dfrac{3\sqrt{32}}{5\sqrt{12\frac{1}{2}}}$.

Find to three decimal places the value of:

25. $4+5\sqrt{3}$.

26. $2-6\sqrt{8}$.

27. $21\sqrt{3}+5\sqrt{27}$.

28. $13\sqrt{91}+2\sqrt{13}$.

29. $\dfrac{2}{\sqrt{6}}$.

30. $\dfrac{5\sqrt{2}}{2\sqrt{5}}$.

31. $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{12}}$.

32. $\dfrac{25\sqrt{2}}{\sqrt{75}-2\sqrt{3}}$.

Express with rational denominators:

33. $\dfrac{\sqrt{x}}{\sqrt{y}}$.

34. $\dfrac{\sqrt{3b^2a}}{15\sqrt{a^2b}}$.

35. $\dfrac{3\sqrt{bx}}{4\sqrt{cx}}$.

36. $\dfrac{a\sqrt{n}}{b\sqrt{m^3}}$.

128. Fractional exponents. Radical expressions may be written in two ways, with *radical signs* or with *fractional exponents.* The relation between the two will now be explained. To do this it is necessary to extend the meaning of the term *exponent,* which, as defined on page 17, applied to positive integral exponents only. We shall assume that the laws which govern the use of integral exponents hold for fractional exponents also.

The fact that $x^2 \cdot x^3 = x^5$ illustrates the more general law

$$x^a \cdot x^b = x^{a+b},$$

where a and b represent either integers or fractions.

Accordingly, $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1$ or x. Since $x^{\frac{1}{2}}$ multiplied by itself gives x, $x^{\frac{1}{2}}$ *must be another way of writing* $\sqrt{x}$.

Hence $\sqrt{x}$ may be written $x^{\frac{1}{2}}$.

Then $4^{\frac{1}{2}} = \sqrt{4} = 2$, and $(25\,a^2)^{\frac{1}{2}} = \sqrt{25\,a^2} = 5\,a$.

Furthermore, $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^1 = x$.

Since $x^{\frac{1}{3}}$ is one of the three equal numbers whose product is x, we may say that $x^{\frac{1}{3}}$ *is another way of writing* $\sqrt[3]{x}$.

Therefore $\sqrt[3]{x}$ may be written $x^{\frac{1}{3}}$.

In general terms, $x^{\frac{1}{n}} = \sqrt[n]{x}$.

Now $x^{\frac{3}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (x^{\frac{1}{2}})^3 = (\sqrt{x})^3$,

and $x^{\frac{3}{2}} = x^{3\ \frac{1}{2}} = (x^3)^{\frac{1}{2}} = \sqrt{x^3}$.

Hence $x^{\frac{3}{2}} = (\sqrt{x})^3$ or $\sqrt{x^3}$.

In general terms, $x \ = \sqrt[n]{x^a}$.

Thus $x^{\frac{a}{n}}$ means the nth root of x to the ath power.

The *numerator* of a fractional exponent denotes the *power* to which the radicand is to be raised, while the *denominator* denotes the *root* to be extracted. Whether one extracts the root first and then raises the result to the power, or vice versa, depends wholly on convenience.

BIOGRAPHICAL NOTE

François Vieta. The reason that algebra is a universal language lies in the fact that the symbols used to indicate the various operations and relations are widely understood and adopted. This has not always been the case, and for a long time during the early history of the subject there was no accepted notation in algebra, but each man used any symbol that suited him. One of the men who did most to establish a fixed notation was François Vieta (1540–1603), a French lawyer who studied and wrote on mathematics as a pastime. He was in public life during his whole career and was well known for his ability to decipher the hidden meaning of dispatches captured from the enemy.

It was he who established the use of the signs + and − for addition and subtraction, which, to be sure, had been used before his time, but were not generally accepted. He also denoted the known numbers in an equation by the consonants, B, C, D, etc., and the unknowns by the vowels, A, E, I, etc. He recognized the existence of negative roots of equations, but rejected them as absurd.

To denote the second and third powers of the unknown, he used the letters Q (*quadratus*) and C (*cubus*) respectively. Instead of using the sign = he wrote aeq. (*aequalis* or *aequatur*). Thus Vieta would have written the equation $x^3 - 8x^2 + 16x = 40$ in the form

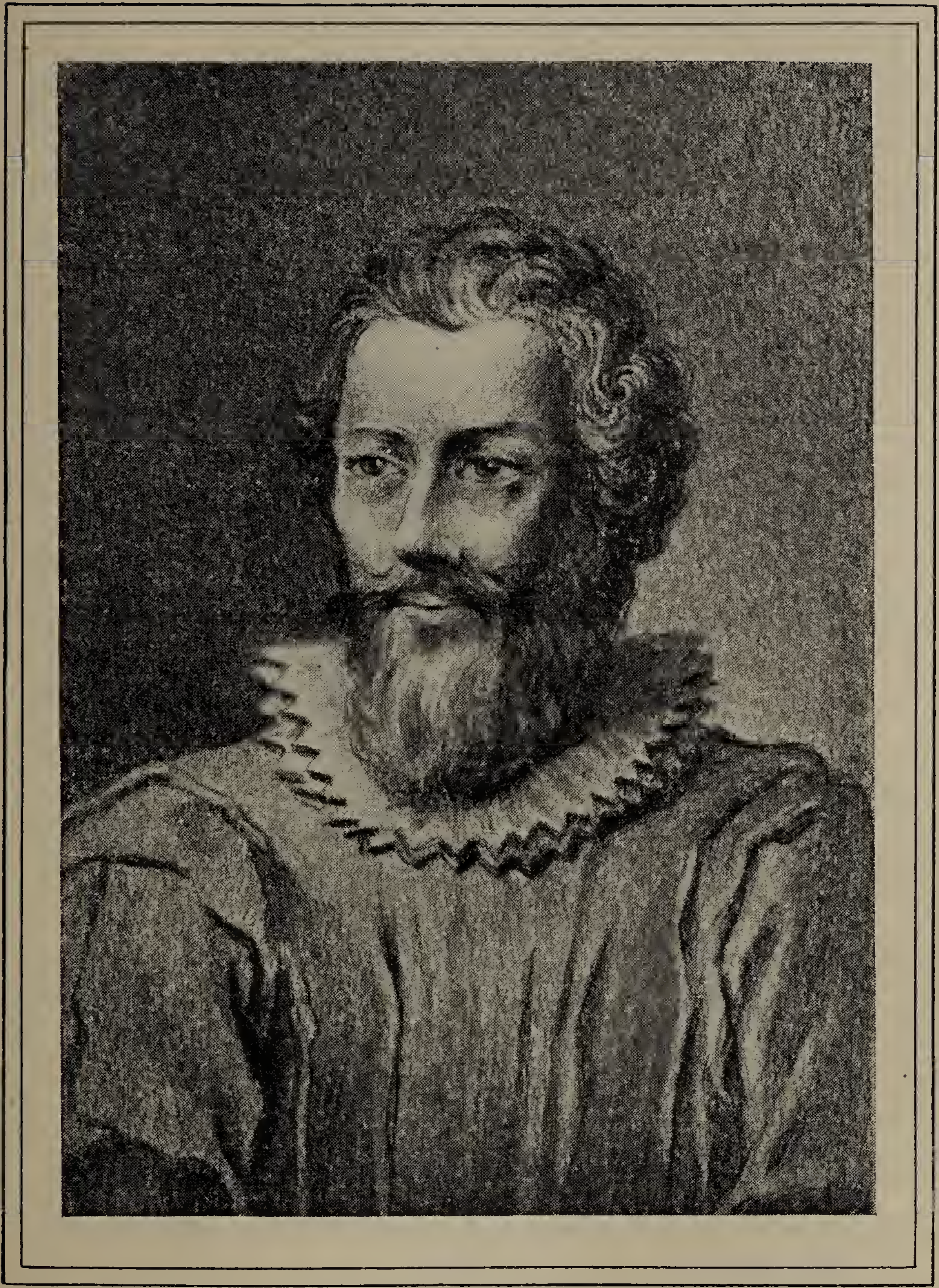
$$1\,C - 8\,Q + 16\,N \text{ aeq. } 40.$$

Before the time of Vieta this equation would have been written in a much more primitive notation. For instance, with writers only a little earlier it would appear as

$$\text{Cubus } \bar{m}\ 8 \text{ Census } \bar{p}\ 16 \text{ rebus aequatur } 40.$$

Operations on equations in this form would certainly be difficult.

Vieta is further distinguished as being the first man to obtain an exact numerical expression for the number π, which occurs in geometry. His form of expression calls for an infinite number of operations which, of course, could never be performed, but the further one proceeded, the closer would be the approximation obtained. In



François Vieta

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a certain sense the familiar sign $\sqrt{\ }$ implies an infinite number of operations, for one can never go through the process of extracting the square root of 2, for instance, and come out even. Vieta's method of denoting π was, however, more involved than this and made use of complicated irrational fractions.

ORAL EXERCISES

Read in radical form:

1. $x^{\frac{1}{2}}$.
2. $x^{\frac{1}{3}}$.
3. $3\,b^{\frac{1}{5}}$.
4. $2\,x^{\frac{2}{3}}$.
5. $5\,y^{\frac{3}{2}}$.
6. $12^{\frac{1}{2}} \cdot y^{\frac{1}{3}}$.
7. $3^{\frac{1}{2}} \cdot x^{\frac{3}{2}}$.
8. $(12\,y)^{\frac{4}{3}}$.
9. $6\,m^{\frac{2}{3}}$.
10. $5\,b^{\frac{5}{4}}$.

Change to fractional exponents:

11. $\sqrt{3}$.
12. $5\sqrt[3]{x}$.
13. $a\sqrt{31}$.
14. $\sqrt{a^2}$.
15. $3\sqrt{x^3}$.
16. $15\sqrt[5]{m^2}$.

Give the numerical value of:

17. $(4)^{\frac{1}{2}}$.
18. $(9)^{\frac{1}{2}}$.
19. $(8)^{\frac{1}{3}}$.
20. $(4)^{\frac{3}{2}}$.
21. $(27)^{\frac{1}{3}}$.
22. $(125)^{\frac{2}{3}}$.
23. $(36)^{\frac{1}{2}}$.
24. $(-64)^{\frac{2}{3}}$.
25. $(\frac{1}{16})^{\frac{3}{2}}$.

EXERCISES

Change to radical form and simplify results:

1. $m^{\frac{3}{2}}$.
2. $2\,x^{\frac{1}{2}}y^{\frac{3}{2}}$.
3. $12\,x^{\frac{2}{3}}y^{\frac{15}{3}}$.
4. $5^{\frac{3}{2}} \cdot am^{\frac{4}{3}}$.
5. $5(rt)^{\frac{3}{2}}$.
6. $(-8)^{\frac{2}{3}} \cdot x^{\frac{5}{2}}$.
7. $c(de)^{\frac{3}{2}}$.
8. $4(16\,x)^{\frac{1}{4}}$.
9. $a^{\frac{2}{b}} \cdot x^{\frac{3}{b}}$.
10. $3\,m^{\frac{3}{2}} \cdot 4\,n^{\frac{5}{2}}$.
11. $12\,x^{\frac{3}{2}} \cdot y^{\frac{5}{4}}$.
12. $6\,a^{\frac{15}{4}} \cdot b^{\frac{12}{5}}$.
13. $(16\,y)^{\frac{3}{2}}$.
14. $5\,m^{\frac{2}{4}}y^{\frac{3}{4}}$.
15. $\left(\frac{1}{m}\right)^{\frac{3}{2}}$.
16. $\left(\frac{2}{x}\right)^{\frac{5}{2}}$.
17. $\left(\frac{12}{5\,y}\right)^{\frac{1}{2}}$.
18. $\left(\frac{2\,x}{3}\right)^{\frac{3}{2}}$.

REVIEW EXERCISES

Evaluate, carrying all inexact answers to two decimal places:

1. $2\sqrt{bm^2} + 3\sqrt{-b^2mx} - 4\sqrt{-bmx}$,

when $b = 1;\ x = 2;\ m = -2.$

2. $2\sqrt{x^2 + \sqrt{5br}}$,

when $x = 4;\ b = \frac{1}{5};\ r = 81.$

3. $5a\sqrt{7x^2 - ay}$,

when $a = 7.2;\ x = 3;\ y = 4.8.$

4. $2x\sqrt{\frac{m^2 - x^2}{a}}$,

when $m = 3;\ x = 2;\ a = 20.$

5. $3m + 2\sqrt{\frac{m}{4} + b}$,

when $m = 1;\ b = 5.$

6. $\frac{a}{\sqrt{b}}$,

when $a = 5;\ b = 7.$

7. $\frac{2\sqrt{x} + \sqrt{y}}{\sqrt{x + y}}$,

when $x = 2;\ y = 3.$

8. $y\sqrt{3r^2 + 2ry + y^2}$,

when $r = 5;\ y = 2.$

9. Express in a form more convenient for computation:

(a) $r = \sqrt{\frac{\pi v^3}{2.5}}$,

(b) $r = \sqrt{\frac{A}{\pi}}$.

Solve for x and y and check:

10. $\dfrac{13x+4}{2}=15.$

11. $\dfrac{x-a}{x-b}=\dfrac{b}{a}.$

12. $\dfrac{1}{ab}+1-\dfrac{ab}{x}=\dfrac{1}{x}.$

13. $x^2+5x-24=0.$

14. $\begin{cases} 3x+5y=13, \\ \dfrac{2x-15y}{3}=9. \end{cases}$

15. $\begin{cases} \dfrac{3}{x}+\dfrac{4}{y}=7, \\ \dfrac{1}{x}-\dfrac{5}{y}=-4. \end{cases}$

16. $4(x^2-2x+1)-9=0.$

17. $\begin{cases} .2x+.5y=-1.1, \\ \dfrac{.3x-.7y}{2}=1.65. \end{cases}$

18. $4x^2+8x+4=0.$

19. Multiply $2(5b^2x^3y)^{\frac{1}{2}}$ by $(3x^2y)^{\frac{1}{2}}$.

Without solving the equations determine whether the pairs of values given below are roots of the corresponding equations:

20. Do $x=2\sqrt{5}$ and $y=3\sqrt{5}$ satisfy

$$x+y=5\sqrt{5}$$

and

$$3x^2-4xy+y^2=65?$$

21. Do $x=2\sqrt{2}$ and $y=5\sqrt{2}$ satisfy

$$\frac{2x+3y}{19}=\sqrt{2}$$

and

$$x^2+5xy-3y^2=-40?$$

22. Do $a=\sqrt{3}$ and $b=\frac{1}{3}\sqrt{3}$ satisfy

$$2a-b=\frac{5}{\sqrt{3}}$$

and

$$\frac{a^2-b^2}{4}=\frac{2}{3}?$$

23. A tree on the bank of a stream is 25 feet from an observer on the opposite bank. The distance from the observer to the top of the tree is 32 feet. If the observer is on a level with the base of the tree, how high is the tree?

24. The altitude of an equilateral triangle is 1 foot. What is the length of one side? What is the area?

25. A man has $200 invested, part at 4% per year and the remainder at 6% per year. The total income from the money is $9 per year. How much money is invested at each rate of interest?

26. The sum of three consecutive numbers is 18. Find the numbers.

27. The sum of the digits of a two-digit number is 7. If 27 is added to the number the result is expressed by the digits in reverse order. What was the original number?

28. A man's age is now five sevenths of what it will be 10 years from now. What is his present age?

29. A field containing 10,000 square feet has a length twice its width. What are the dimensions of the field?

30. The altitude of a triangle is two thirds the base. If the area is 24 square feet, what are the base and altitude?

CHAPTER XXII

QUADRATIC EQUATIONS

129. Introduction. The simplest quadratic equation is one in which the term of the first degree is lacking; as, for example, $x^2 - 4 = 0$ or $x^2 + 41 = 0$, where the terms in x are lacking. In order to solve these equations one term should be transposed (if necessary) so as to bring one term on each side of the equation. The square root of each member should be taken, not neglecting the ± sign before the square root of the constant term. Equations like the foregoing are sometimes called *pure quadratics*.

ORAL EXERCISES

Solve the following pure quadratic equations:

1. $x^2 - 9 = 0.$
2. $x^2 - 25 = 0.$
3. $y^2 - 16 = 0.$
4. $x^2 - 1 = 0.$
5. $y^2 - 4 = 0.$
6. $-x^2 + 4 = 0.$
7. $y^2 = 1.$
8. $y^2 - 64 = 0.$
9. $x^2 - 4 = 12.$
10. $-y^2 + 16 = 0.$
11. $x^2 - 4a^2 = 0.$
12. $y^2 - 25b^4 = 0.$
13. $x^2 - 16a^2 = 0.$
14. $x^2 - 9c^2 = 0.$
15. $y^2 - 25b^4 = 0.$
16. $25 = m^2.$
17. $4a^2 - 9x^2 = 0.$
18. $-169 + x^2 = 0.$

Before taking up the work that follows, the student should review the method of forming trinomial squares given on page 152.

ORAL EXERCISES

What terms should be added in order to make the following expressions perfect trinomial squares?

1. $x^2 + 2x + ?$
2. $y^2 + 6y + ?$
3. $x^2 - 4x + ?$
4. $y^2 - 2y + ?$
5. $x^2 + 16x + ?$
6. $x^2 - 8x + ?$
7. $x^2 + 3x + ?$
8. $r^2 + r + ?$
9. $x^2 - \frac{1}{2}x + ?$
10. $x^2 + 5x + ?$
11. $y^2 - 9y + ?$
12. $x^2 + \frac{2}{3}x + ?$
13. $x^2 + \frac{2}{5}x + ?$
14. $x^2 + \frac{3}{4}x + ?$
15. $t^2 - \frac{3}{2}t + ?$

130. Solution by completing the square. The method of solving quadratic equations by completing the square depends on the rearrangement of the terms of the equation so that the left-hand member is a trinomial and a perfect square and the right-hand member is a constant.

In order that the left-hand member may become such a perfect square, it is usually necessary to add a properly chosen number to each member of the equation.

EXAMPLES

1. Solve $x^2 + 2x - 15 = 0$. (1)

Solution. Transposing, $x^2 + 2x = 15$. (2)

Adding 1 to each member of (2),

$$x^2 + 2x + 1 = 16. \tag{3}$$

Then $$(x + 1)^2 = 4^2. \tag{4}$$

Extracting the square root of each member of (4),

$$x + 1 = \pm 4. \tag{5}$$

Whence $$x = -1 + 4 = 3$$

and $$x = -1 - 4 = -5.$$

Check. Substituting 3 for x in (1), $9 + 6 - 15 = 0$, or $0 = 0$.

Substituting -5 for x in (1), $25 - 10 - 15 = 0$, or $0 = 0$.

2. Solve $2x^2 + 3x - 2 = 0.$ (1)

Solution. Transposing, $2x^2 + 3x = 2.$ (2)

Dividing (2) by the coefficient of x^2,

$$x^2 + \tfrac{3}{2}x = 1. \qquad (3)$$

Adding $\left(\tfrac{3}{4}\right)^2$ to each member of the equation,

$$x^2 + \tfrac{3}{2}x + \left(\tfrac{3}{4}\right)^2 = \tfrac{9}{16} + \tfrac{16}{16}. \qquad (4)$$

Then $$\left(x + \tfrac{3}{4}\right)^2 = \left(\tfrac{5}{4}\right)^2. \qquad (5)$$

Extracting the square root, $x + \tfrac{3}{4} = \pm\tfrac{5}{4}.$ (6)

$$x = -\tfrac{3}{4} \pm \tfrac{5}{4}$$

and $$x = \tfrac{1}{2} \text{ or } -2.$$

Check. Substituting $\tfrac{1}{2}$ for x in (1),

$$2\left(\tfrac{1}{4}\right) + 3\left(\tfrac{1}{2}\right) - 2 = 0$$

$$\tfrac{1}{2} + \tfrac{3}{2} - 2 = 0, \text{ or } 0 = 0.$$

Substituting -2 for x in (1),

$$2(-2)^2 + 3(-2) - 2 = 0.$$

$$8 - 6 - 2 = 0, \text{ or } 0 = 0.$$

The method of solving a quadratic equation in x illustrated in the preceding examples may be stated in the

Rule. *Transpose terms so that the terms containing* x *are in the first member and those which do not contain* x *are in the second.*

Divide both members of the equation by the coefficient of x^2 *unless that coefficient is* $+1$.

Then add to both members the square of half the coefficient of x *(in the equation just obtained), thus making the first member a perfect trinomial square.*

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation and write the sign ± before the square root of the second member, thus obtaining two linear equations.

Solve the equation in which the second member is taken with the sign + and then solve the equation in which the second member is taken with the sign −. The results are the roots of the quadratic.

CHECK. *Substitute each result separately in place of* x *in the original equation. If the resulting equations are not obvious identities, simplify until they become so.*

Before attempting to solve quadratic equations by the method of completing the square, it is often desirable to observe whether they can be solved by factoring according to the method set forth on page 172, since it is frequently simpler to solve a quadratic equation by factoring than to solve it by the method of completing the square.

EXERCISES

Solve by completing the square, and check as directed by the teacher:

1. $x^2 + 6x - 7 = 0.$
2. $x^2 + 4x = 12.$
3. $y^2 - 10y = 24.$
4. $a^2 + 8a - 105 = 0.$
5. $m^2 - 2m = 80.$
6. $x^2 - 6x = 16.$
7. $2x^2 - 7x + 5 = 0.$
8. $x(x + 4) - 3(x + 4) = 0.$
9. $2x^2 - 3x + 1 = 0.$
10. $2x^2 + x - 10 = 0.$
11. $3x^2 + 15x + 10 = 3x - 2.$
12. $3y^2 + 7y = -4.$
13. $y^2 - 13 = 0.$
14. $5x^2 + 13x - 6 = 0.$
15. $6x^2 - 5x + 1 = 0.$
16. $5x^2 - x - 4 = 0.$
17. $x^2 = 15.$
18. $6x^2 - 5x - 6 = 0.$
19. $4x^2 + 13x + 9 = 0.$
20. $5y^2 + 22y = 15.$
21. $21x^2 - 2x - 8 = 0.$
22. $20x^2 - 43x + 14 = 0.$
23. $12m^2 - 67m = 50.$
24. $30y^2 - y - 1 = 0.$
25. $135x^2 - 42x = 16.$
26. $42x^2 - 51x + 15 = 0.$

27. Why is not equation (5), Example 1, page 326, written with the sign ± before each member?

28. $3x^2 - 7x + 1 = 0.$ (1)

Solution. Transposing, $3x^2 - 7x = -1,$ (2)

Dividing each member of (2) by 3,

$$x^2 - \tfrac{7}{3}x = -\tfrac{1}{3}. \tag{3}$$

Adding $(-\tfrac{7}{6})^2$ to each member of (3),

$$x^2 - \tfrac{7}{3}x + \tfrac{49}{36} = \tfrac{49}{36} - \tfrac{12}{36} = \tfrac{37}{36}. \tag{4}$$

Then

$$(x - \tfrac{7}{6})^2 = \tfrac{37}{36}. \tag{5}$$

$$x - \tfrac{7}{6} = \pm \tfrac{1}{6}\sqrt{37}. \tag{6}$$

$$x = \tfrac{7}{6} \pm \tfrac{1}{6}\sqrt{37} \tag{7}$$

$$= \frac{7}{6} \pm \frac{6.0828}{6} \tag{8}$$

$$= \frac{13.0828}{6} \text{ or } \frac{.9172}{6} \tag{9}$$

$$= 2.180 \text{ or } 0.153. \tag{10}$$

Check. Since the values in (10) are not *exact* values of x, they will, if substituted for x in (1), make its first member *nearly* but not exactly equal to zero.

An exact check on the radical forms of the roots can be obtained by substituting from equation (7) in equation (1). This check may be shortened by substituting both roots at the same time, as follows:

Substituting $\tfrac{7}{6} \pm \tfrac{1}{6}\sqrt{37}$ for x in $3x^2 - 7x + 1 = 0,$

we get

$$3(\tfrac{7}{6} \pm \tfrac{1}{6}\sqrt{37})^2 - 7(\tfrac{7}{6} \pm \tfrac{1}{6}\sqrt{37}) + 1 = 0,$$

$$3(\tfrac{49}{36} \pm \tfrac{14}{36}\sqrt{37} + \tfrac{37}{36}) - 7(\tfrac{7}{6} \pm \tfrac{1}{6}\sqrt{37}) + 1 = 0,$$

$$\tfrac{49}{12} \pm \tfrac{14}{12}\sqrt{37} + \tfrac{37}{12} - \tfrac{49}{6} \mp \tfrac{7}{6}\sqrt{37} + 1 = 0.$$

The radical terms vanish because the two upper signs before them must be taken together, and then the two lower signs.

Therefore, $\frac{49}{12} + \frac{37}{12} - \frac{98}{12} + \frac{12}{12} = 0,$

or $\frac{98}{12} - \frac{98}{12} = 0.$

In quadratic equations like the preceding the radical forms of the roots are often sufficient; at other times values to two or three decimal places are necessary. Unless otherwise directed, obtain only the radical forms of irrational roots.

In the following exercises obtain correct to three decimal places the values of any radical answers which may occur:

29. $x^2 - 5x + 2 = 0.$
30. $\frac{3}{4} - x^2 = 3x.$
31. $3x + 5x^2 = 14.$
32. $4x + 15 = 25x^2.$
33. $6t^2 - 5t = 21.$
34. $5x^2 - 20x = -19.$
35. $2x^2 - 15 = x^2 - 2x - 7.$
36. $x^2 = 21x - 5.$
37. $14x^2 + 5x = 2.$
38. $3z^2 + \frac{2}{3}z - 2 = 0.$
39. $x^2 + 3.4x + 1.7 = 0.$
40. $x^2 - 2.1x - .6 = 0.$
41. $.2x^2 + .7x - .42 = 0.$
42. $.5x^2 - 2.3x + .4 = 0.$
43. $x^2 - \frac{4\sqrt{3}}{3}x = 5.$
44. $\frac{a}{3} + 4 - \frac{15}{a} = 0.$
45. $\frac{x}{5} - \frac{2}{3x} - 1 = 0.$
46. $\frac{2}{t-2} - \frac{3t}{2} = 0.$
47. $x = \frac{3}{x+2}.$
48. $\frac{4}{y+5} = -3y.$
49. $\frac{1+x}{2+x} - \frac{1}{6} = \frac{3-x}{5-x}.$
50. $\frac{1}{2x+4} = \frac{15}{4} + \frac{1}{x-2}.$
51. $\frac{y+2}{y-4} + \frac{y-1}{y+5} + 1 = 0.$

BIOGRAPHICAL NOTE

Karl Friedrich Gauss. Standing in the very front rank of mathematicians, with Archimedes and Newton, is Karl Friedrich Gauss (1777–1855). He was the son of a bricklayer and was afforded an education, much against the will of his parents, by a nobleman who had noticed his remarkable talents.

At the age of nineteen he had made discoveries which had baffled all mathematicians up to that time, and continued to make important contributions to the theories of mathematics and astronomy for the next fifty years.

The student has undoubtedly noticed that quadratic equations have two roots. It is not difficult to prove that cubic equations have three roots and that equations of the fourth degree have four roots. That any equation in one unknown has a number of roots equal to its degree was first proved by Gauss. He gave three distinct proofs of this fact, although no one before his time had been able to prove it at all.

In collaboration with another professor at the University of Göttingen, he invented the telegraph independently of the American, S. F. B. Morse, and probably earlier.

REVIEW EXERCISES

Solve, and check as directed by the teacher:

1. $10x^2 - 11x + 1 = 0.$

2. $11x^2 - 5x = 7.$

3. $x + 2 = 25x^2.$

4. $9x^2 = 15x + 2.$

5. $(4x - 3)(x + 2) = (3x + 25)(2x + 3).$

6. $(3x + 4)(x - 1) = (5x + 2)(3x - 2).$

7. $5x^2 + 11x - 12 = 0.$

8. $\frac{4x}{3} = 5 + \frac{25}{x}.$

9. $\frac{21}{x} = x + 2.$

10. $\frac{91x}{17} - 3 = \frac{22}{17x}.$

11. $\frac{b}{5} = \frac{10}{b} + 12.$

12. $\frac{x^2}{x+2} = \frac{5}{4}.$

13. $\frac{x}{3}=\frac{9+2x}{x}$. 14. $\frac{2x}{x+1}+\frac{10x}{15}=25$. 15. $\frac{3x^2}{x+1}=\frac{5x}{3}$.

16. $(14x-5)(3x+2)=(21x+5)(3x-2)$.

17. $\frac{x+1}{x-5}+\frac{x+2}{x-3}=5$.

18. $\frac{5x-3}{5x+2}=\frac{1}{2}-\frac{2x-1}{3x+4}$.

19. $\frac{x}{5-x}=\frac{3}{4}+\frac{3}{x}$.

20. $y=\frac{21}{2y+3}$.

21. $2x^2-5x+2=0$.

22. $\frac{13x^2}{5x+1}=\frac{2x}{13}+2$.

23. $\frac{2x+1}{5x+3}+\frac{1}{12}-\frac{x+1}{x-1}=0$.

24. $\frac{x+2}{5x-1}-\frac{1}{15}=\frac{x+2}{x+1}$.

PROBLEMS

(Reject all answers which do not satisfy the conditions of the problems.)

1. The square of a certain number added to three times the same number is equal to 40. What is the number?

2. One half the square of a number is equal to six more than twice the number. Find the number.

3. What are the two consecutive numbers whose product is 812?

4. What are the two consecutive odd numbers whose product is 483?

5. What are the two consecutive even numbers whose product is 2208?

6. What are the three consecutive numbers such that the product of the last two is equal to the sum of all three?

7. The area of a rectangular field is 1 square mile. The length is 400 feet greater than the width. Find the dimensions.

8. Two square fields have a total area of 10,000 square feet. The side of one field is 50 feet longer than the side of the other. What are the dimensions of the two fields?

9. The diagonal of a rectangle is 50 feet. If one side is 20 feet longer than the other, what are the dimensions?

10. One side of a right triangle is two thirds of the other. The hypotenuse is 28 feet. Find the length of the two sides.

11. The same number expresses both the perimeter of a square in feet and its area in square feet. Find the side of the figure.

12. An automobile made a trip of 125 miles out and 125 miles back in 11 hours. The average speed on the return trip was 3 miles an hour less than on the outward trip. What was the rate in each direction?

13. Two trains run 384 miles. The faster train has a speed 4 miles per hour greater than the speed of the other and requires 1 hour less time. What is the speed of each train?

14. The edges of two cubical boxes differ by 2 inches. The volumes differ by 56 cubic inches. Find the dimensions of the boxes.

15. Two pumps can fill a tank in 45 minutes. One pump takes 10 minutes longer to fill it alone than the other one does. What is the time that each pump alone requires to fill the tank?

16. If the price of eggs is raised 25 cents per dozen, $3 will buy 24 fewer eggs than would have been obtained at the original price. What was the original price?

17. A rectangular garden is made up of a flower-bed 10 feet longer than it is wide, surrounded by a walk 3 feet wide. The area of the bed is 204 square feet less than the area of the total garden. Find the dimensions of the bed and of the whole garden.

The velocity of a body falling from rest varies as the expression $V = 32\,t$, where V is the velocity in feet per second and t is the time in seconds that the body has been falling. The distance traveled by the same body is expressed by the equation $S = 16\,t^2$, where S is the distance traveled in feet and t, as before, is the time in seconds.

18. How long will it take for a stone dropped from an elevation of 2100 feet to reach the ground?

19. What will be the velocity of a body which has fallen a distance of 1000 feet, neglecting the resistance of the air?

20. A man drops a stone down a well and hears the splash 2.59 seconds later. If the sound is heard .09 seconds after the splash is seen, how deep is the well? How fast does the sound travel in feet per second?

History of the quadratic equation. Though the development of the method of solving quadratic equations is closely connected with the general growth of algebra, yet it is possible to indicate rather briefly the most important steps in the process.

The first writer on formal algebra was Diophantos, who lived at Alexandria, in Egypt, about A. D. 275. Most of his work that is preserved is devoted to the solution of problems that lead to equations. So far as we know he was the first to indicate the unknown number by a single letter, in this respect being far in advance of many mathematicians who lived much later. It is a little remarkable, in fact, that so able and original a man as Diophantos should have exerted so little influence on his successors. He solved his quadratic equations by a method not unlike that of completing the square, but his imperfect knowledge of the nature of numbers made it impossible for him to understand the entire significance of the process. Though he made every effort not to consider equations whose roots were not positive integers, sometimes they would creep in, and under such circumstances, when his method led him to a negative or irrational root, he rejected the whole equation as absurd or impossible. Even when both the roots were positive he took only the one afforded by the positive sign in the formula for solving a quadratic.

The difficulties of Diophantos are typical of those encountered by mathematicians for the next fifteen hundred years. The difficulty lay not in finding a formal method of solving the equation but in understanding the result after it was obtained. The meanings of negative and of imaginary numbers (that is, even roots of negative numbers) have been two of the most difficult of all mathematical ideas for men to grasp.

Five or six hundred years later the Hindus devised a general solution of the quadratic, but their chief advance over Diophantos lay in the fact that they did not regard an equation whose roots were negative as necessarily absurd, but merely rejected the negative result with the remark, "It is inadequate; people do not approve of negative roots." The Hindus, however, did realize that a quadratic equation sometimes has two roots, a fact that Diophantos never comprehended.

No material gain in the understanding of the solutions of the quadratic can be found until the seventeenth century. The keenest mathematicians of the sixteenth century, like Cardan and Vieta, rejected negative roots, though by this time irrational roots were admitted. In fact, in 1544 Stifel, a German, published an algebra in which irrational numbers are included among the numbers proper. But he affirms that except in the case where a quadratic equation has two positive roots no equation has more than one root. It was not until the work of Descartes and Gauss became widely known that the nature of the roots of all kinds of quadratic equations was completely understood.

131. Systems involving a linear and a quadratic equation.

A *quadratic equation* in two unknowns contains one or more terms of the second degree, but no term of higher degree in those unknowns. Every system of equations in two unknowns in which one equation is *linear* and the other *quadratic* can be solved by the method of *substitution*.

Example

Solve the system
$$\begin{cases} 2\,y^2 + x^2 = 6, & (1) \\ 3\,x - y = 5. & (2) \end{cases}$$

Solution. Solving (2) for y in terms of x,

$$y = 3\,x - 5. \tag{3}$$

Substituting $3\,x - 5$ for y in (1),

$$2(3\,x - 5)^2 + x^2 = 6. \tag{4}$$

From (4), $$19\,x^2 - 60\,x + 44 = 0. \tag{5}$$

Solving (5), $$x = 2 \text{ or } \tfrac{22}{19}. \tag{6}$$

Substituting 2 for x in (3),

$$y = 1.$$

Substituting $\frac{22}{19}$ for x in (3),

$$y = -\tfrac{29}{19}.$$

The two sets of roots are therefore $x = 2$, $y = 1$, and $x = \frac{22}{19}$, $y = -\frac{29}{19}$.

Check. Substituting 2 for x and 1 for y in (1) and (2),

$$2 + 4 = 6,$$

$$6 - 1 = 5.$$

Substituting $\frac{22}{19}$ for x and $-\frac{29}{19}$ for y in (1) and (2),

$$\tfrac{1682}{361} + \tfrac{484}{361} = \tfrac{2166}{361} = 6,$$

$$\tfrac{66}{19} + \tfrac{29}{19} = \tfrac{95}{19} = 5.$$

The similarity between this method and that for the solution of linear systems by substitution should be carefully noted.

The method given in the preceding solution for solving a system consisting of one quadratic and one linear equation is stated in the

RULE. *Solve the linear equation for one unknown in terms of the other.*

Substitute this value in the quadratic equation and solve the resulting equation.

Substitute each of the roots of the quadratic equation thus found in the linear equation, and solve, thus obtaining two sets of roots of the simultaneous system.

CHECK. *Substitute as usual, in both equations.*

EXERCISES

Solve the following systems, pair results, and check each set of roots:

1. $2x^2+y=33,\quad x+y=5.$
2. $x^2+y^2=13,\quad 3x+2y=12.$
3. $x^2-y=1,\quad y-x=1.$
4. $x+y^2=19,\quad 3x+y=13.$
5. $x^2-5y=4,\quad x-5y=-38.$
6. $2x^2+5y^2=125,\quad x+3y=15.$
7. $4m-n^2=-28,\quad n-7m=-8.$
8. $5x^2+y^2=30,\quad x+2y=11.$
9. $6A^2+7B^2=82,\quad 5A-4B=7.$
10. $x^2+5xy=14,\quad x+2y=4.$
11. $x^2+5xy=76,\quad x-y=1.$
12. $x^2+3xy-2y^2=188,\quad 2x=5y.$
13. $5x^2+2xy+3y^2=195,\quad 2x+y=13.$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problems.)

1. The sum of two numbers is 5. The sum of their squares is 125. Find the numbers.

2. The difference of two numbers is 25. The sum of their squares is 925. What are the numbers?

3. A rectangular playing field is 180 feet longer than it is wide. Its area is 49,500 square feet. Find the dimensions.

4. The area of one square field is nine sixteenths that of another. The perimeter of one is 20 feet less than that of the other. What are the dimensions?

5. A park is a right triangle. One of the two shorter sides is 25 feet longer than the other. The area is 625 square feet. What are the dimensions?

6. The annual income from a certain investment is $200. If the principal were $500 less and the interest rate 1% more, the income would be $25 more. What was the amount of the principal and what was the rate of interest?

7. The perimeter of a rectangle is 26 inches, and the area is 42 square inches. Find the dimensions.

8. The diagonal of a rectangle is 25 inches and its perimeter is 70 inches. What are its dimensions?

9. The area of a rectangle is 150 square feet. The perimeter of the figure is 50 feet. Determine the dimensions.

10. The speed of a boat is twice the speed of the current in a river. The boat heads straight across the stream and at the end of 5 minutes is .93 miles from its starting point. What is the speed of the boat and of the current?

11. The perimeter of a rectangle and the area are both expressed by the number 18. What are the dimensions?

12. A motorist leaves a point A and travels north. At the same time a second motorist, who travels 50% faster than the first, leaves a point 3 miles east of A and travels east. Half an hour later the distance between them is 17 miles. Find the rate of travel of each motorist.

REVIEW EXERCISES

Factor the following expressions:

1. $6x^2 + 13xy + 6y^2$.

2. $8x^2 + 12x + 2ax + 3a$.

3. $2mny + 6mnx + 9mx^2 + 3mxy + 9nx^2 + 3nxy + 6n^2x + 2n^2y$.

Solve for the unknowns involved and check:

4. $x^2 - 5 = 0$.

5. $a^2 + 3a = 5$.

6. $3m^2 - 2m - 15 = 0$.

7. $3x + 2y - 5 = 14,$
$25x^2 + 2xy + 3y^2 = 657.$

8. $3x + 5y = -7,$
$2x + 3y - 25 = -29.$

9. $\frac{2x}{3} + \frac{5y}{2} = -8\frac{2}{3},$
$\frac{3x}{4} - \frac{2y}{5} = 3\frac{1}{10}.$

10. $.5x + .23y = .61,$
$.33y - .54x = 3.39.$

Divide:

11. $10x^4 + 31x^3y + 26x^2y^2 - xy^3 - 6y^4$ by $x + y$.

12. $14x^2y^2 - 3xy^3 + 2y^4 + 15$ by $x - 3y$.

Find the square root of the following expressions:

13. 25,403.25.

14. 37,281.05.

15. 200.453.

16. $4 - 12a + 9a^2 + 2(2x - 3ax - 6ay + 4y) + x^2 + 4xy + 4y^2$.

17. $4x^2 + 12xy + 16x + 9y^2 + 24y + 16$.

18. A workman has a sawhorse with legs each 30 inches long, arranged like an inverted V. The horse stands 24 inches high. The man finds that he can adjust the height of the horse by varying the distance between its legs. How much must he decrease the distance between the legs to raise the horse 2 inches?

19. A man standing with his feet together is 6 feet tall. When he stands with his feet 2 feet apart his apparent height is 5.83 feet. How long are his legs?

20. The perimeter of a rectangle is 14 feet and its diagonal is 5 feet. What is its area?

21. A collection of dimes and quarters is worth $6.55. There are one more than eight times as many quarters as dimes. How many coins of each kind are there?

22. Two trains start from points 100 miles apart and travel toward each other until they meet. One train has a speed of 40 miles per hour and the other has a speed of 35 miles per hour. What are the distances between the meeting point and the points from which they originally started?

CHAPTER XXIII

REVIEW OF FIRST YEAR ALGEBRA

FUNDAMENTAL OPERATIONS

132. Algebraic expressions, order of fundamental operations, and terms. Review section 4, page 5, section 14, page 20, and the definitions on pages 16, 48, and 51.

EXERCISES

Simplify:

1. $8 - 3 + 10 - 4$.

2. $7 + 15 \div 3 - 8$.

3. $5 \cdot 6 \div 2 + 7$.

4. $30 \div 3 \div 5$.

5. $40 \div 4 \div 2 + 2 - 3 + 4 \cdot 2$.

6. $(21 - 3 \cdot 4)(17 - 5 \cdot 2) - 3 \cdot 7$.

Find the numeric value of:

7. $3x + 4 \div 2 - x$ if $x = 4$.

8. $r^2 - 5r + 6$ if $r = 3$; if $r = 5$; if $r = 1$; if $r = 12$.

9. $y^3 - 3y^2 + 4y - 20$ if $y = 0$; if $y = 1$; if $y = 4$.

10. Does $3(5x - 2) + 14 = 4x^2 + 2x + 11$ if $x = 3$? if $x = 0$? if $x = 2$?

11. Does $\dfrac{(m-1)(m+1)}{m^2 + 2m + 1} = \dfrac{m-1}{m+1}$ if $m = 3$?

133. Addition and subtraction. Review rules and examples of addition and subtraction on pages 36, 37, 39, 48, 50, 51, 68, and 70.

EXERCISES

Add:

1. $12, -8, +4, -6, +7$.

2. $7x, 3x, -4x, 3x, -5x$.

3. $3r - 7w$, $4r - 3w$, $6r + 5w$, and $-3r + 4w$.

4. $5x^2 - 6xy + 3y^2$, $-7x^2 + 3xy - 4y^2$, $3x^2 + 3xy + y^2$, and $-x^2 + xy$.

5. What name is given to each 2 in $r^2 - 2r$? Distinguish between a coefficient and an exponent.

6. What is the coefficient of $a + b$ in $2x(a + b)$? in $5(a + b)d$? in $3a + 3b$? in $ac + bc$?

Write so that a, b, or x shall have a polynomial coefficient:

7. $ax + bx$.

8. $3a - 4ab$.

9. $a + 3ay + a$.

10. $ax + bx + cx$.

11. $2a - a + 3ay$.

12. $2px - 3rx + x - 5x$.

Write the following so that the binomial will have a binomial coefficient:

13. $(a - 2)x + (a - 2)y$.

Solution. $(a - 2)x + (a - 2)y = (a - 2)(x + y)$.

14. $(3a - 2b)3x - (3a - 2b)2y$.

15. $4r(5a - 4c) - 3s(-4c + 5a)$.

16. $(3x - 4y)8a^2f - (-4y + 3x)7bc^4$.

Subtract the first term from the second in the following:

17. $7ax$, $9ax$.

18. $33ay^8$, $15ay^8$.

19. $-6x^4z$, $18x^4z$.

20. $7a^4b$, $-7a^4b$.

21. $5x^2 - y^2$, $7x^2 - y^2$.

22. $7mp + 2rv$, $3rv - 6mp$.

23. $3x - 4y + 7z$, $5x + y - 6z$.

Find the expression which added to the first term will give the second:

24. $3m^2 - 5m + 3$, $6m^2 - 11m + 7$.
Check for $m = 3$.

25. $9x^2 - 7bx + b^2$, $12x^2 + 9bx - 8b^2$.
Check for $x = 2$, $b = 3$.

26. $7b^2 + 3bm - 3m^2$, $10b^2 - 4bm - 5m^2$.

27. From the sum of $5ax^2 + 7by^3 - z^4$, and $6ax^2 - 3by^3 - 2z^4$, take $7ax^2 + 3by^3 - 3z^4$.

28. From the sum of $3x^2 + 2xy - 5y^2$, $2x^2 - 5y^2$, and $x^2 - 10xy + 3y^2$, take the sum of $-5x^2 + 2xy + y^2$, $10x^2 + 2xy$, and $9x^2 + 5xy + 3y^2$.

134. Multiplication. Review rules and examples of multiplication on pages 41, 90, 91, 93, and 95 and review the discussion on arrangement of polynomials on pages 96–97.

EXERCISES

Multiply:

1. $(3x - 5)(2x + 6)$.

2. $(3r^2 - 3r + 5)(r^2 - 5r + 6)$.

3. $(m^4 + 2m^3 - 4)(m^4 - 2m^2 - 3)$.

4. $(n^3 - n^2 + 7)(n^4 + 3n^3 - 2n^2)$.

5. $(2b^2 - 3bc + c^2)(b^2 + 5bc - 4c^2)$.
Check with $b = 2$, $c = 3$.

6. $(n^2 - nr + r^2)(n^2 - r^2 + nr)$.

HINT. Write the second factor as $n^2 + nr - r^2$, that is, in descending powers of n.

7. $(n^2 - np + p^2)(n^2 + p^2 + np)$.

8. $(t^3 - t^2 + t)(at^2 + a + at)$.
Check for $t = 3$, $a = -1$.

9. $(4h^2 + 6k^2 + 9hk)(4h^2 + 6k^2 - 9hk)$.
Check for $h = 1$, $k = 2$.

10. $(at^2 - 2at^3 + 5a)(6at^4 - 2at - 4at^2)$.

11. $(m^4 + m^3 + 2m^2 - m - 1)(m - 1 + 2m^2)$.

12. $(x^n + 3x^m + 2)(x^n - x^m - 2)$.

13. $\left(2 + \frac{3m}{4} - \frac{2m^2}{3}\right)\left(1 - \frac{4m}{5} + \frac{m^2}{6}\right)$. Check for $m = 2$.

14. $(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$.

15. $(5x - 9x^5 + x^2 - 4x^4 - 3x^3 + 10)(-x^3 + 3x - x^2 + 2x^4)$.

16. $(x^4 - x^3 + 25 - 5x - 4x^2)(x + x^2 + 5)$.

17. $(x^{2a} + x^4 + x^2 + x^{a+2} - x^{a+1} + x^3)(x^a - x^2 + x)$.

18. Does $15(x - a) - 6(x + a) = 3(5a - 3x)$ if $x = 2a$?

19. Does $ax(a + 3) + a(10 - a^2) = x + 2$ if $x = a - 3$?

135. Division. Review pages 43, 109, and 110 for rules, examples, and oral exercises in division and rules and examples on pages 114–115 for the division of polynomials.

EXERCISES

Perform the indicated division:

1. $(2m^2 - 5m + 3) \div (2m - 3)$.
Check for $m = 3$.

2. $(x^2 - 7x + 12) \div (x - 3)$.
Check with $x = -2$.

3. $(6m^2 + 10mr - 4r^2) \div (m + 2r)$.

4. $(18r^2 + 69rt - 165t^2) \div (9r - 15t)$.

5. $(8m^3 - 10m^2 - 13m + 15) \div (2m - 3)$.

6. $(6a^3 + 6a^2 - 28 - 26a) \div (2a + 4)$.

7. $(6x^6 - 5x^4 + 25x^3 - 17x^5) \div (5x^2 - 2x^3)$.

8. $(s^6 - 7s^3 - 8) \div (s^2 + 4 + 2s)$.
Check with $s = 2$.

9. $(x^3 - 5a^2x + 2a^3) \div (x^2 + 2ax - a^2)$.

10. $(2x^4 - 12x^2 - 2 + 11x - 7x^3) \div (1 - 3x - 2x^2)$.

11. $(23s^2 - 13s^3 + 2s^4 - 60 - s) \div (5 + 3s - s^2)$.

12. $(m^3n^3 + 8p^3 + 125 - 30mnp) \div (mn + 2p + 5)$.

13. $(x^3 + y^3 + z^3 - 3xyz) \div (x + y + z)$.

14. $(32x^4 - 60 - 2x - 104x^3 + 92x^2) \div (5 + 6x - 4x^2)$.

15. $(a^2 - 2ab + b^2 - 9x^2) \div (3x + b - a)$.

16. $(m^4n^4 - 2m^2n^2p^2 + p^4 - 16q^4) \div (m^2n^2 - p^2 + 4q^2)$.

17. $(8b^2c^2 + 1 - 16c^4 - b^4) \div (4c^2 - b^2 - 1)$.

18. $(m^5 + m^3 + 8m^2 + 8) \div (m^2 - 2m + 4)$.

19. $(3r^4 - 23r^3 - 7r^2 + 42r - 10) \div (3r^2 + r - 5)$.

20. $(2x^5y^{10} - 9x^4y^8 + 15x^3y^6 - 27x^2y^4 + 20xy^2 - 4)$
$\div (2x^2y^4 - 7xy^2 + 2)$.

21. $(m^6 - 7m^4r + 3m^2r^2 - 21r^3) \div (m^4 + 3r^2)$.

22. $(x^{4a+4} - 16x^4) \div (x^{a+1} + 2x)$.

23. $(a^{4n} + a^4 + a^{2n+2}) \div (a^2 + a^{2n} - a^{n+1})$.

136. Parentheses. Review the rules and examples of pages 83–89.

EXERCISES

Remove parentheses and simplify:

1. $2 - 1 - (3 - 1) + (2 - 3)$.
2. $5 + 6 - (-5 + 3) + (-6 - 2) - 3 + 10$.
3. $a + (a - b) - (a - 3b)$.
4. $a - (a - m) + (2m - 3a)$.
5. $8x + (3r - 8x + 2) - (2r - 3x + 2)$.
6. $x - y + 3(x - y) - 4(2x - y)$.
7. $4x - a + [-(3c - x) - (2a - 3x)]$.
8. $m - [-(m - 4) + (3r - 2m) - 6m] + 7r$.
9. $x - 4 - (s - 2x) - [3(s - x + 4) - 2(6 - 4s)]$.
10. $4t^2 - 3t - 2t(3 + 2t)$.
11. $x^2 - 4 - (x - 1)(x + 4)$.
12. $4x^2 - 3r^2 - (2r - 3x)(2x - r)$.
13. $4x - 2(x - 3) - 3[x - 3(4 - 2x) + 8]$.
14. $(x - 4)(x - 3) - (x - 3)(x + 2)$.
15. $5a^2 - [6a^2 + 2a - (3a^2 - 1) + 10a] + 5(2a - 1)$.
16. $(x + 4)(x^2 - 4x + 16) - (x - 4)(x^2 + 4x + 16)$.

137. Important special products. Review the examples of special products on pages 134–143. In addition to the four special products (I–IV) there given, three special products follow which should be studied with care.

I. The square of the polynomial $a + b - c$ gives the formula

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$$

This expressed verbally is:

The square of any polynomial is equal to the sum of the squares of each of the terms plus twice the algebraic product of each term by every term that follows it in the polynomial.

II. The cube of the binomial $a + b$ gives the formula

$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

This expressed verbally is:

The cube of the sum of two numbers equals the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

III. Similarly,

$$(a - b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3.$$

This can be expressed verbally in a manner similar to II.

ORAL EXERCISES

State the results of the indicated multiplication:

1. $(x + 3)^2$.
2. $(3\,h + 1)^2$.
3. $(3\,m + 2)^2$.
4. $(a - 3)^2$.
5. $(3\,x - 5)^2$.
6. $(8\,s - .3)^2$.
7. $(x + 7\,a)^2$.
8. $(2\,x - 3\,y)^2$.
9. $(x^2 - x)^2$.
10. $(a^2 + 3\,a)^2$.
11. $(3\,r - 4\,s)^2$.
12. $(mn - 3\,qr)^2$.
13. $(.5\,x + y)^2$.
14. $(.6\,x + .2\,y)^2$.
15. $(5\,x + .03\,y)^2$.
16. $(a^x + 1)^2$.

Solution. $(a^x + 1)^2 = a^{2x} + 2\,a^x + 1.$

17. $(x^x + y^b)^2$.
18. $(x^{2b+1} + 3)^2$.
19. $(x^{3a+1} + y^b)^2$.
20. $(x - 5)(x + 5)$.
21. $(2\,x - 1)(2\,x + 1)$.
22. $(ax + 3)(ax - 3)$.
23. $(4\,r - 3\,s)(4\,r + 3\,s)$.
24. $(x + 3)(x + 4)$.
25. $(x + 1)(x + 7)$.
26. $(r - 3)(r - 5)$.
27. $(m - 3)(m + 4)$.
28. $(x + 2)(x - 9)$.
29. $(ax - 3)(ax + 5)$.
30. $(2\,x - 1)(2\,x + 3)$.

31. $(4n - 2)(4n + 3)$.
32. $(a^2 - 3a)(a^2 + 4a)$.
33. $(a + b + c)^2$.
34. $(a + b - x)^2$.
35. $(a - b + x)^2$.
36. $(r + s + 1)^2$.
37. $(r - s + 1)^2$.
38. $(a + c + 2)^2$.
39. $(a + c)^3$.
40. $(a + 3)^3$.
41. $(a - x)^3$.
42. $(x + 1)^3$.
43. $(1 - a)^3$.
44. $(h + 2r)^3$.

EXERCISES

Find the following products:

1. $[(x + y) + 1][(x + y) - 1]$.
2. $[(x + a) + 3][(x + a) - 3]$.
3. $[(x - r) + 3][(x - r) - 3]$.
4. $[(m + 3) + x][(m + 3) - x]$.
5. $[(2a - b) + c][(2a - b) - c]$.
6. $(x - 3r + 8)(x + 3r - 8)$.
7. $[(h + s) + (a + b)][(h + s) - (a + b)]$.
8. $(.5x + 7y)(.5x - 7y)$.
9. $(1.2h + 1.3s)(1.2h - 1.3s)$.

PROBLEMS

1. A square field has a side $2s$ feet. What is the decrease in the area if its side is decreased 6 feet?

2. From each corner of a square piece of tin of side c inches a square of side x inches is cut. By turning up the sides an open box is formed. Show that $c^2 - 4x^2$ inches is the area of the box.

3. Express the area $c^2 - 4x^2$ inches of Problem 2 as the product of two binomials.

4. Using the results of Problem 3 find by a short method the area of the inside of the box if $c = 15$ and $x = 2$; if $c = 80$ and $x = 5$; if $c = 70$, $x = 15$.

5. The dimensions of a rectangular box are n, $n + 2$, $n + 5$. Express (a) the sum of the edges, (b) the total outer surface, (c) the volume of the box.

6. An entrance hall with a center fountain is in the shape of a hollow square, as in Figure 1. What is the area of the floor surface?

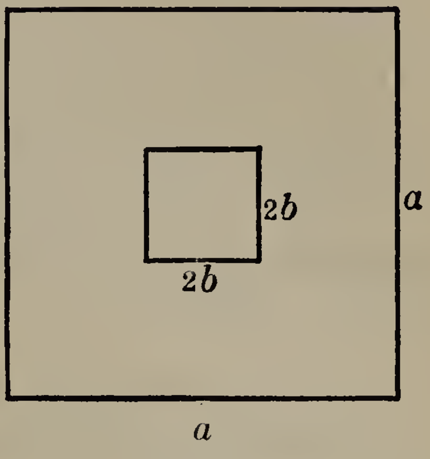


FIG. 1

7. Find the area of the hall in Problem 6 if $a = 9$ yards and $b = 2$ feet.

8. Show that the area of Figure 2 illustrates the product $a^2 - b^2$.

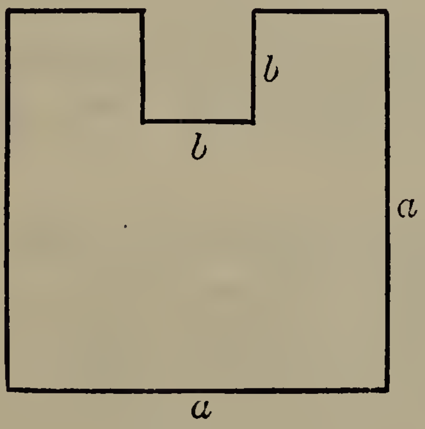


FIG. 2

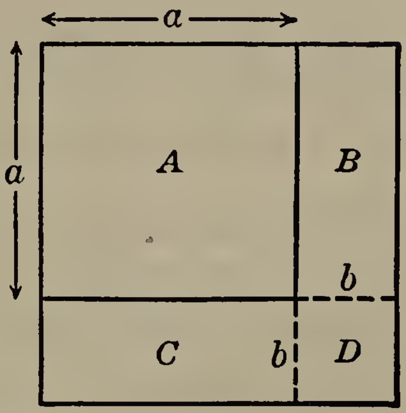


FIG. 3

9. What product is illustrated by the area of A in Figure 3?

10. Find the total area of Figure 3.

11. Find the area of Figure 3 if $a = 10$, $b = 6$; if $a = 45$, $b = 8$.

12. If two equal boxes of dimensions b, $b - 3$, and $b + 7$ are placed end to end, find (*a*) the sum of the outer edges, (*b*) the outer surface, and (*c*) the combined volume.

13. The formula for changing a Centigrade thermometer reading to Fahrenheit is $F = \frac{9}{5}C + 32$.

What is the Fahrenheit reading when the Centigrade reading is (*a*) 55°? (*b*) 18°? (*c*) 75°? (*d*) 100°? (*e*) 32°? (*f*) $-15°$?

14. The area of a circle is given by the formula πr^2, in which $\pi = 3.1416$ and r equals the radius of the circle. What is the area of a circular flower bed of radius 5 feet? of radius 7 feet? of radius r feet?

15. What is the area of a circular ring left after cutting a circle of radius r from a circle of radius R?

16. What is the area of a cross-section of pipe whose diameter is 12 inches, if the inside diameter is 8 inches?

17. What is the area of a cross-section of a concrete standpipe whose outside diameter is 30 feet, if the masonry is 4 feet in thickness?

18. The volume of a cylinder is found by using the formula $\pi r^2 h$, in which r equals the radius of the circular base and h is the height of the cylinder. What is the volume of a cylinder if r equals 10 and $h = 15$?

19. What is the volume of water contained in the standpipe of Problem 17 if $h = 200$ feet?

How many gallons would the standpipe hold, allowing $7\frac{1}{2}$ gallons to the cubic foot?

20. The surface of a cylinder is $2\pi r(h + r)$, in which r is the radius of the base and h is the height of the cylinder. Find the amount of tin in 10 cases of cylindrical cans which have a circular base of radius 2 inches and a height of 4 inches. (A case contains 144 cans.)

21. Find the total surface of a cylinder of radius $a + 7$, and height $a - 4$. Use $\pi = \frac{22}{7}$.

22. Find the volume of a cylinder of radius $r - 7$ and height $r + 7$.

23. The formula for the surface of a sphere is $4\pi r^2$. What is the surface of a sphere of radius 5 inches? 7 inches? 10 feet?

24. The dome of St. Peter's, Rome, is in the form of a hollow hemisphere 140 feet in diameter (inside measurements). How many square yards of gold foil would be needed to cover the inside of the dome?

25. What is the volume of a sphere of 1 foot radius? of 5 foot radius? ($V = \frac{4}{3}\pi r^3$.)

26. How many spherical balls of $\frac{1}{2}$ inch radius can be made from a spherical ball of lead of radius 12 inches?

27. What is the volume of a sphere whose radius is $2r$ inches? $r - 7$ inches?

LINEAR EQUATIONS IN ONE UNKNOWN

138. Definitions. Review the definitions and illustrations of an equation and of an axiom on pages 55–59; of an identity, of an equation of condition, and of the root of an equation on page 72.

Each of the axioms is used in the solution of the

EXAMPLE

Solve $3x - \frac{2}{3} = \frac{10}{3} + 2x$.

Solution. $3x - \frac{2}{3} = \frac{10}{3} + 2x$ (1)

C

Multiplying (1) by 3, (Ax. III)

$$9x - 2 = 10 + 6x \tag{2}$$

Adding 2 to each member of (2), (Ax. I)

$$9x = 12 + 6x \tag{3}$$

Subtracting $6x$ from each member of (3), (Ax. II)

$$3x = 12 \tag{4}$$

Dividing (4) by 3, (Ax. IV)

$$x = 4 \tag{5}$$

Check. Substituting 4 for x in (1), we have

$$12 - \frac{2}{3} = \frac{10}{3} + 8 \text{ or } \frac{34}{3} = \frac{34}{3}$$

Since substituting 4 for x satisfies (1), 4 is the root of (1).

139. Transposition. Review the definitions and illustrations of transposition on pages 74–75.

140. Equivalent equations. Two or more equations in one unknown, even if of very different form, are **equivalent** if all are satisfied by every value of the unknown which satisfies any one of them.

Equations (2), (3), (4), and (5) of section 138 are each equivalent to equation (1) and to each other, for all are satisfied by the same value of the unknown, $x = 4$.

Of the four axioms or assumptions of section 36 we shall make constant use. If the "same number" referred to in each is expressed arithmetically, the result is always an equation *equivalent* to the *original* one. Further, if *identical expressions involving the unknown* be added to or subtracted from each member of an equation, the resulting equation is equivalent to the first. If, however, both members of an equation be multiplied by identical expressions containing the unknown, the resulting equation *may not* be equivalent to the original one.

Multiplying each member of the equation $x - 5 = 2$ by $x - 2$, we get $x^2 - 7x + 10 = 2x - 4$, or $x^2 - 9x + 14 = 0$. Now this last equation has the roots 2 and 7, whereas the given equation has the root 7 only. Here the root 2 was introduced by multiplying the given equation by $x - 2$. Results obtained from the use of Axiom III with multipliers which contain an unknown should always be carefully checked. When a root is obtained which does not satisfy the original equation, this root should be rejected.

When the divisor contains the unknown, the use of Axiom IV may result in the loss of a root which the process of checking will not discover.

For example, if each member of $x^2 - 25 = x + 5$ is divided by $x + 5$, the result is $x - 5 = 1$, whence $x = 6$. But $x = -5$ also satisfies $x^2 - 25 = x + 5$. The root -5 was lost by dividing by $x + 5$.

If *both members of an equation are divided by a factor containing an unknown,* this factor should be set equal to zero. The root thus obtained is a root of the given equation.

With these and with certain other rare exceptions which will be noted later, the application of the axioms will produce an equation equivalent to the given one.

For solving equations in one unknown which do not involve fractions we have the

Rule. *Free the equation of any parentheses it may contain.*

Transpose and solve for the unknown involved.

Reject all values for the unknown which do not satisfy the original equation.

Checking the solution of an equation is often called testing or verifying the result. For this we have the

RULE. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify each member of the resulting identity until the two members are seen to be identical.*

If the correct substitution of the root for the unknown does not satisfy the equation, an error has been made in the solution.

EXERCISES

Solve and check:

1. $5x + 1 = 2x + 7$.
2. $1 + r + 5r + 17 = 0$.
3. $6x - 10 = 2x + 14$.
4. $2k - 3 = 5k + 12$.
5. $15x = 3(2x - 5)$.
6. $3(2x - 1) - (5x - 1) = 0$.
7. $6(4t - 5) - 11(2t - 3) = 0$.
8. $4(x - 2) + 3(2 - x) - 3x = 6(x + 1)$.
9. $(x + 1)(x - 2) = x^2 + 3$.
10. $x^2 - (x - 1)(x + 2) = 4x + 7$.
11. $2r(r - 3) = (2r + 5)(r - 3)$.
12. $(x + 4)^2 - (2 - x)^2 = 84$.
13. $(1 - x)(x + 2) + (x + 3)(x + 4) = 2$.
14. $(x + 4)(x + 3) = (x + 2)(x + 1) + 42$.
15. $5(x - 3) + 3(8 - x) + 29 = 7(6 - x) + 50$.
16. $(k - 3)^2 + 3(k - 4)^2 = 4(k - 5)^2 - 3$.

Solve for x or y:

17. $x - 2a = 4a - x$.
18. $c - x = x - c$.

19. $8s - x = x - 4r.$

20. $ax - 2ab = 4ab - ax.$

21. $2klh - kh^2 = kl^2 - klx + khx.$

22. $ax - a^2 + 5a = 6 + 3x.$

23. $6c^2 - 12c + 4x = c^3 - c^2x + 4cx - 8.$

24. $a^3 - c^3 = ax - cx.$

141. Solution of problems. Review the steps in the solution of a verbal problem as stated and illustrated on pages 62–63.

PROBLEMS

1. The sum of two consecutive numbers is 1175. Find the numbers.

2. The sum of four consecutive odd numbers is 1120. What are the numbers?

3. The product of two consecutive even numbers increased by 4119 equals the product of the next two consecutive odd numbers.. Find the numbers.

4. What number must be added to 8, 9, 11, and 15 in order that the product of the first and third may be 147 less than the product of the second and fourth?

5. One number exceeds another number by 7. The square of the larger number exceeds the square of the smaller number by 119. What are the numbers?

6. One pupil is four years older than another. Eight years from now the first will be $\frac{6}{5}$ as old as the second. Find their ages now.

7. Two men are 50 and 30 years of age respectively. How many years ago was the older twice the age of the younger?

8. One man is twice as old as another. The sum of their ages 12 years ago was $\frac{1}{2}$ the sum of their ages 12 years hence. How old are they now?

9. If each side of a square is increased by 9 feet, the area of the square will be increased by 171 square feet. What is the area of the square?

10. A certain rectangle is 10 feet longer than it is broad. If it were 3 feet shorter and 6 feet wider, its area would be 51 square feet greater. What are its length and width?

11. A rectangle is 17 feet narrower and 5 yards longer than a certain square. The area of the square is 675 square feet greater than the area of the rectangle. Find the dimensions of the rectangle.

12. A certain square grass plot has a strip 6 feet wide taken from all sides for a walk. The area of this border is 816 square feet. What is the side of the square?

13. A sum of $8.50 is in dimes, nickels, and quarters. There are three more dimes than nickels, and 9 more quarters than dimes and nickels together. How many coins of each kind are there?

14. A sum of $23.40 consists of dollars, quarters, and dimes. If there are 6 more dimes than dollars, and the number of quarters is 4 less than twice the number of dimes, find the number of coins of each kind.

15. The sum of the digits of a certain two-digit number is 15. If the order of the digits is reversed, the number is decreased by 9. Find the number.

16. The digits of a certain three-digit number beginning at the left are consecutive odd numbers. If the sum of the digits is 21, find the number.

17. One angle of a triangle is 3 times another. The third angle is 20° more than the sum of the other two angles. If the sum of the angles of a triangle is 180°; how many degrees are there in each angle?

18. The sum of twice one acute angle of a right triangle and 4 times the other acute angle equals 330°. How many degrees are there in each acute angle?

19. The formula for the area of a circle is $A = \pi r^2$. ($\pi = \frac{22}{7}$.) If the radius of a circle is decreased by 7 inches, the area is decreased 770 square inches. Find the original radius.

20. By subtracting 2 inches from the radius of a circle whose radius is 8 inches, how much is the circumference decreased? the area?

21. If a circular hoop 1 foot longer than the circumference of the earth is placed about the earth so that it is everywhere equidistant from the equator and lies in its plane, how far above the equator will the hoop be?

22. Compare the result of Problem 21 with the one obtained when a similar process is carried out with a sphere 24 inches in diameter, instead of with the earth.

23. Two trains start from points 270 miles apart. What is the speed of each if one travels 2 miles an hour faster than the other and they meet in 5 hours?

24. An automobile averaging 24 miles per hour leaves 6 hours before an express train going 45 miles an hour in the same direction as the automobile. How long after it starts will the train overtake the automobile?

25. A marksman heard a bullet strike a target 1375 feet distant $5\frac{1}{2}$ seconds after he fired. If the velocity of sound is 1100 feet per second, what was the average velocity of the bullet?

FACTORING

142. Definition of factoring. *Factoring* is the process of finding the two or more arithmetic or algebraic expressions whose product is equal to a given expression.

In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience in multiplication and division is called upon to enable us to determine the factors.

143. Rational expressions. A *rational* algebraic expression is one which can be written without the use of indicated roots of the letters involved.

Thus 3, $2a$, $5x-\sqrt{3}$, and r^2 are rational expressions. In this chapter factors which involve radicals will not be sought.

144. Integral expressions. If a rational expression can be written so as not to involve an indicated division in which an unknown letter occurs in a denominator, it is said to be *integral.*

Thus 2, $5k$, $\frac{a}{3}$, and $2x+7$ are integral expressions.

145. Prime factors. An integral expression is *prime* when it is the product of no two rational integral expressions except itself and 1.

It must be remembered that to factor an integral expression means to resolve it into its prime factors.

The methods of this chapter enable one to factor integral rational expressions in one letter which are not prime, as well as some of the simpler expressions in two letters. No attempt is made even to define what is meant by prime factors of expressions which are not rational and integral.

There is no simple operation the performance of which makes us sure that we have found the prime factors of a given expression. Only insight and experience enable us to find prime factors with certainty.

A partial check that may be applied to all the exercises in factoring consists in multiplying together the factors that have been found. If the result is the original expression, correct factors have been found, though they may not be prime factors.

146. Polynomials with a common monomial factor. The type form is

$$ab + ac + ad.$$

Factoring, $ab + ac + ad = a(b + c + d)$.

ORAL EXERCISES

Factor:

1. $2a + 4$. **4.** $2c - 6c^2$. **7.** $ax^2 - a^2x$.

2. $5x + 15$. **5.** $9x^2 - 3x$. **8.** $4cx - 8c^2$.

3. $a^2 + a$. **6.** $8x^3 - 4x^2$. **9.** $5ax - 2ax^3$.

10. $14h - 21h^2k$. **12.** $3r + 6rs + 9r^2$.

11. $2c + 4c^2 - 2cd$. **13.** $4a - 10a^2 - 2a^3$.

147. Polynomials which may be factored by grouping terms and taking out a common binomial factor. The type form is

$$ax + ay + bx + by.$$

Factoring,

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (x + y)(a + b). \end{aligned}$$

EXERCISES

Separate into polynomial factors:

1. $2(a + b) + x(a + b)$. **3.** $2a(x - y) + b(x - y)$.

2. $3(b + 4) + a(b + 4)$. **4.** $a(c - d) - b(c - d)$.

5. $h(m + 3r) - 2k(m + 3r)$.

6. $2r(5x - 4a) - 9s(5x - 4a)$.

7. $r(x - s) + y(s - x)$.

HINT. Write in the form $r(x - s) - y(x - s)$, etc.

8. $2a(c - 3h) + b(3h - c)$.

9. $5r(3m - 2s) - 2t(2s - 3m)$.

10. $ad + 2dx + 3ar + 6xr$.

11. $akr + ahr - ahs - aks$.

12. $4a^2 - 4ax - ac + cx$.

13. $mr - 2r + ms - 2s + mt - 2t$.

14. $2ax - bx + cx + 2ay - by + cy$.

HINT. Group thus $(2ax - bx + cx) + (2ay - by + cy)$, etc.

15. $3r^{3x} - r^{2x} + 3r^x - 1$.

16. $a^{3m} + a^{2m} + a^m + 1$.

148. Trinomials which are perfect squares. The type form is

$$a^2 \pm 2ab + b^2.$$

Factoring, $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

ORAL EXERCISES

Factor:

1. $a^2 + 2ax + x^2$. **3.** $x^2 + 4x + 4$.

2. $x^2 - 2xy + y^2$. **4.** $r^2 - 10rs + 25s^2$.

5. $9 + 6a + a^2$.

6. $16 - 8ax + a^2x^2$.

7. $16r^2 - 24r + 9$.

8. $9x^2 - 12xy + 4y^2$.

9. $r^2 - 12rs + 36s^2$.

10. $a^{2n} - 12a^n + 36$.

11. $(r - s)^2 - 6(r - s) + 9$.

12. $9 + 6(a + x) + (a + x)^2$.

13. $16 - 8(a - 2x) + (a - 2x)^2$.

14. $x^{2a} - 4x^ay^b + 4y^{2b}$.

15. $.09 + .6(a + b) + (a + b)^2$.

149. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

Factoring, $a^2 - b^2 = (a + b)(a - b)$.

More generally,
$$\begin{aligned} &a^2 + 2ab + b^2 - c^2 + 2cd - d^2 \\ &= a^2 + 2ab + b^2 - (c^2 - 2cd + d^2) \\ &= (a + b)^2 - (c - d)^2 \\ &= (a + b + c - d)(a + b - c + d) \end{aligned}$$

EXERCISES

Factor:

1. $a^4 - x^8$.

2. $x^4y^4 - z^4$.

3. $(a - 2)^2 - c^2$.

4. $4(x + 3)^2 - y^2$.

5. $16(x - y)^2 - z^2$.

6. $a^2 - (b + c)^2$.

7. $4r^2s^2 - (r - s)^4$.

8. $(a - c)^2 - (d + e)^2$.

9. $a^2(r + 2s)^2 - (x - y)^2$.

10. $4(a - x)^2 - 9a^2(c - 2d)^4$.

11. $r^2 + 2rs + s^2 - (x^2 - 2xy + y^2)$.

12. $x^2 - y^2 + 2yz - z^2$.

13. $16h^2 - 25k^2 + 70kl - 49l^2$.

14. $x^2 + 6x + 9 - y^2 + 2yz - z^2$.
15. $4r^2 + 9s^2 - 16t^2 - 25m^4 - 12rs + 40tm^2$.
16. $121x^8 - 1 - 18y - 81y^2$.
17. $a^2 - b^2 - (a - b)$.
18. $m + n - m^2 + n^2$.
19. $x^2 - 4y^2 + x - 2y$.
20. $r^2s^2 - 4s^2 - m^2r^2 + 4m^2$.
21. $m^4x^4 - m^4 - 81x^4 + 81$.

22. $x^{2n} - y^{6m}$.
23. $1.44x^4 - 1.21y^6$.
24. $\frac{1}{9}a^4 - \frac{9}{16}y^2$.
25. $.0009r^4 - .0025s^2$.
26. $1.69a^2b^4 - 1.96c^2$.
27. $\frac{1}{16}s^6 - \frac{1}{49}r^4$.

150. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

Review the explanations and examples on pages 161–162.

EXERCISES

Factor:

1. $r^2 - r - 90$.
2. $r^2 - 3r - 18$.
3. $x^2 - 2x - 24$.
4. $9 - 10x + x^2$.
5. $r^2 - 4rs + 3s^2$.
6. $m^2 + 8m - 20$.
7. $k^2 + 19k - 20$.
8. $k^2 - 21k + 20$.
9. $m^2 - 7mn + 10n^2$.
10. $r^2 - 6rs + 9s^2$.
11. $(x - y)^2 + 4(x - y) + 3$.
12. $x^4 - 8x^2y^2 + 16y^4$.

Solution. $x^4 - 8x^2y^2 + 16y^4 = (x^2 - 4y^2)^2$
$= (x + 2y)^2(x - 2y)^2$.

13. $x^8 - 17x^4 + 16$.
14. $a^{2n} + 12a^n + 35$.
15. $a^{4x} - 5a^{2x} - 6$.
16. $b^{4y} - 2b^{2y} - 35$.
17. $s^{4x} - 10s^{2x} + 9$.

151. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

Review the examples and rule on pages 164–165.

EXERCISES

Factor:

1. $2x^2 + 5x + 2$.
2. $2a^2 + 9a + 10$.
3. $3r^2 + 13r + 12$.
4. $3r^2 - 8r + 5$.
5. $4x^2 - 13xy + 10y^2$.
6. $10x^2 - 29x + 10$.
7. $12x^2 - 11xy + 2y^2$.
8. $3r^2 + r - 2$.
9. $2a^2 - a - 15$.
10. $12m^2 - 25mn + 12n^2$.
11. $12r^2 + 45rs - 12s^2$.
12. $3 - r - 10r^2$.
13. $10r^2 - 19rs - 15s^2$.
14. $2x^2 + 5xy - 12y^2$.
15. $6a^{2n} - 7a^n + 2$.
16. $3a^{4n} - 10a^{2n} - 8$.
17. $10x^{2n} - x^n - 3$.
18. $.72r^2 + 7rs + 12s^2$.
19. $.15r^2 - .24rs - .63s^2$.
20. $\frac{1}{2}r^2 - \frac{7}{12}rs + \frac{1}{6}s^2$.

152. Expressions reducible to the difference of two squares. The type form is

$$a^4 + ka^2b^2 + b^4.$$

If k has such a value that the trinomial is not a perfect square, a trinomial of this type can often be written as the *difference of two squares*. Thus, if $k = 1$, the addition and subtraction of a^2b^2 accomplishes this result.

EXAMPLES

1. Factor $a^4 + a^2b^2 + b^4$.

Solution.
$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab). \end{aligned}$$

2. Factor $25m^4 + 26m^2n^2 + 9n^4$.

Solution. If $4m^2n^2$ is added, the expression becomes a perfect trinomial square. Adding and subtracting $4m^2n^2$, we have

$$\begin{aligned} 25m^4+26m^2n^2+9n^4 &= 25m^4+30m^2n^2+9n^4-4m^2n^2 \\ &= (5m^2 + 3n^2)^2 - (2mn)^2 \\ &= (5m^2+3n^2+2mn)(5m^2+3n^2-2mn). \end{aligned}$$

EXERCISES

Factor:

1. $x^4 + x^2y^2 + y^4$.
2. $r^4 + r^2s^2 + s^4$.
3. $a^4 + a^2b^4 + b^8$.
4. $r^4 + 3r^2s^2 + 4s^4$.
5. $m^8 + m^4 + 1$.
6. $x^4 - x^2y^2 + 16y^4$.
7. $x^4 - 12x^2y^2 + 16y^4$.
8. $25a^4 - 19a^2 + 1$.
9. $25x^4 - 11x^2 + 1$.
10. $16r^4 - 17r^2s^4 + s^8$.
11. $4r^4 - 44r^2s^2 + 49s^4$.
12. $25x^4 - 19x^2 + 9$.
13. $81a^4 + 11a^2b^2 + 4b^4$.
14. $36a^4 - 25a^2b^2 + 4b^4$.
15. $4a^4 + 1$.

HINT. $4a^4 + 1 = 4a^4 + 4a^2 + 1 - 4a^2$.

16. $c^4 + 4d^4$.
17. $64a^4x^2 + x^2$.
18. $4a^{4n} + b^{4n}$.
19. $x^{4a+4} + 4y^{8a+4}$.

153. A binomial the sum or the difference of two cubes. The type form is

$$a^3 \pm b^3.$$

Dividing $a^3 + b^3$ by $a + b$ gives the quotient $a^2 - ab + b^2$, and $a^3 - b^3$ divided by $a - b$ gives the quotient $a^2 + ab + b^2$.

Therefore $$a^3 + b^3 = (a + b)(a^2 - ab + b^2), \qquad (1)$$

and $$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \qquad (2)$$

Formulas (1) and (2) above may be applied as in the

EXAMPLES

1. Factor $h^3 + 8$.

Solution. $$\begin{aligned} h^3 + 8 = h^3 + 2^3 &= (h + 2)(h^2 - 2h + 2^2) \\ &= (h + 2)(h^2 - 2h + 4). \end{aligned}$$

2. Factor $27 - y^3$.

Solution. $$\begin{aligned} 27 - y^3 = 3^3 - y^3 &= (3 - y)(3^2 + 3y + y^2) \\ &= (3 - y)(9 + 3y + y^2). \end{aligned}$$

EXERCISES

Factor:

1. $m^3 - 64$.

2. $c^3 + b^6$.

3. $r^3 - s^9$.

4. $(2a)^3 - (3b)^3$.

5. $8r^3 - 27s^3$.

6. $125x^3 + 8y^6$.

7. $216 - r^3$.

8. $m^6 + n^6$.

9. $(a + b)^3 + c^3$.

10. $m^6 - r^6$.

Solution.

$$\begin{aligned} m^6 - r^6 &= (m^3 + r^3)(m^3 - r^3) \\ &= (m + r)(m^2 - mr + r^2)(m - r)(m^2 + mr + r^2). \end{aligned}$$

11. $x^6 - y^6$.

12. $a^6 - 64$.

13. $1 - a^6$.

14. $a^{12} - b^6$.

15. $x^{12} - 1$.

16. $x^6 - 64y^6$.

17. $a^3 + b^3 + a + b.$

Solution.
$$\begin{aligned} a^3 + b^3 + a + b &= (a^3 + b^3) + (a + b) \\ &= (a + b)(a^2 - ab + b^2) + (a + b) \\ &= (a + b)(a^2 - ab + b^2 + 1). \end{aligned}$$

18. $x^3 - y^3 + x - y.$

19. $m^3 - n^3 - m + n.$

20. $x^3 - 8y^3 + x - 2y.$

21. $r^3 + 8(a + b)^3.$

22. $r^3s^6t^9 - 64k^{15}.$

23. $x^3 - \frac{1}{r^3}.$

24. $(a + b)^3 + (a - b)^3.$

25. $a^{3n} + b^{3n}.$

26. $27m^{3r+3} + 64r^{6x+9}.$

154. The Remainder Theorem. If any rational integral expression in x be divided by $x - n$, the remainder is the same as the original expression with n substituted for x. This fact is illustrated in the

EXAMPLE

Divide $x^2 - 7x + 12$ by $x - n$.

Solution.
$$\begin{array}{r|l} & x + (n - 7) \\ \hline x - n & x^2 - 7x + 12 \\ & \underline{x^2 - nx} \\ & (n - 7)x + 12 \\ & \underline{(n - 7)x \qquad - n^2 + 7n} \\ & \quad \text{Remainder} = n^2 - 7n + 12 \end{array}$$

Here the remainder $n^2 - 7n + 12$ is the same as $x^2 - 7x + 12$, the given expression, when n is substituted for x.

EXERCISES

1. Divide $x^2 + bx + c$ by $x - n$ and show that the remainder is $n^2 + bn + c$.

2. Divide $x^2 + bx + c$ by $x - a$ and find the remainder.

3. Divide $x^3 + ax^2 + bx + c$ by $x - n$ and find the remainder.

4. In $(x^3 + x^2 - 5x + 3) \div (x - 2)$, find the remainder (*a*) by division, (*b*) by the Remainder Theorem.

Solution. (*b*) $2^3 + 2^2 - 5 \cdot 2 + 3 = 5.$

5. In $(x^3 - x + 5) \div (x - 3)$, find the remainder (*a*) by division, (*b*) by the Remainder Theorem.

By use of the Remainder Theorem find the remainders in the following:

6. $(x^3 + x^2 - 5x + 8) \div (x - 3)$.
7. $(x^3 - 3x - 15) \div (x + 4)$.
8. $(r^3 - 2r^2 - 100) \div (r - 5)$.
9. $(s^3 - 2s^2 - 2s - 3) \div (s + 3)$.
10. $(r^4 - 2r^3 + r - 2) \div (r - 2)$.
11. $(r^4 - 3r^2 + 2r - 1) \div (r + 2)$; $\div (r - 1)$.
12. $(2m^4 - 4m^3 + m^2 - 4m + 4) \div (m - 1)$.

155. Factor Theorem. By substituting 3 for x in $x^2 - 7x + 12$ we obtain $9 - 21 + 12$, or 0. Thus the Remainder Theorem shows, without actually performing the division, that $x - 3$ will divide $x^2 - 7x + 12$ without a remainder; that is, divide it exactly. Again, if 4 is substituted for x in $x^2 - 7x + 12$, the expression equals zero. Hence $x - 4$ is a factor of $x^2 - 7x + 12$. These examples illustrate the

THEOREM. *If any rational integral expression in* x *becomes zero when a number* n *is substituted for* x, *then* x − n *is a factor of the expression.*

The Factor Theorem may be used to factor some of the preceding exercises and, in addition, many others which are very difficult to factor by previous methods.

NOTE. By means of the Factor Theorem we are able to solve cubic and higher equations when the roots are integers. The solution of the general cubic equation is one of the famous problems of mathematics and one which is accompanied by many interesting applications. This problem was first solved by the Italian, Tartaglia, about 1530, but was published by Cardan, to whom Tartaglia explained his solution on the pledge that he would not divulge it. For many years the credit for the discovery was given to Cardan, and to this day it is usually called Cardan's Solution.

When searching for the values of x which will make an expression zero, only integral divisors of the last term of the expression (arranged according to the descending powers of x) need be tried, for the last term of the factor must be an integral divisor of the last term of the expression.

EXAMPLE

Factor $x^3 + 4x + 5$.

Solution. If $x - n$ is a factor of $x^3 + 4x + 5$, then n must be an integral divisor of 5. Now the factors of $+5$ are $1, -1, 5$, and -5. If -1 is put for x, then $x^3 + 4x + 5$ equals zero, hence $x + 1$ is a factor of $x^3 + 4x + 5$. Dividing $x^3 + 4x + 5$ by $x + 1$, we obtain the quotient $x^2 - x + 5$. Since $x^2 - x + 5$ is prime, the factors of $x^3 + 4x + 5$ are $x + 1$ and $x^2 - x + 5$.

ORAL EXERCISES

1. Is $(x - 1)$ a factor of $x^3 + 3x - 4$?
2. Is $(x - 2)$ a factor of $2x^3 + x^2 - 20$?
3. Is $(a - 2)$ a factor of $a^3 - 3a + 2$?
4. Is $(x - 1)$ a factor of $x^3 + 3x^2 - 4$?

5. Is $(r+1)$ a factor of $r^3 - 4r^2 - 4r + 1$?
6. Is $(r-3)$ a factor of $2r^3 - r^2 + 5$?
7. Is $(s+1)$ a factor of $3s^3 - 5s^2 + 8$?
8. Is $(m-3)$ a factor of $2m^3 - 5m^2 - 9$?

EXERCISES

Factor:

1. $x^3 + x - 2$.
2. $x^3 + 2x + 3$.
3. $a^3 + a^2 - 36$.
4. $x^3 + x - 10$.
5. $r^3 + r^2 - 12$.
6. $s^3 - 2s^2 - 5s + 6$.
7. $r^3 - r^2 + 4r - 4$.
8. $x^3 - x^2 - 4$.
9. $y^3 - y^2 - 9y + 9$.
10. $x^3 + 2x^2 + 3x - 6$.
11. $r^3 - 7r + 6$.
12. $r^3 - 4r^2 + 4r - 16$.
13. $x^4 - 7x^2 - 6x$.
14. $x^3 - 7x^2 + 4x + 12$.
15. $2x^3 - 2x^2 - x - 6$.
16. $x^3 - x^2 - 48$.

156. The sum or difference of two like powers. The type form is

$$a^n \pm b^n.$$

The cases in which $a^n \pm b^n$ is divisible by $a + b$ or $a - b$ can be determined by the Factor Theorem.

Thus in $a^n - b^n$, where n is either an odd or an even integer, substitute b for a. Then $a^n - b^n$ becomes $b^n - b^n = 0$. Therefore $a - b$ is always a factor of $a^n - b^n$.

In $a^n - b^n$, n being even, put $-b$ for a. Then $a^n - b^n$ becomes $b^n - b^n = 0$, since $(-b)^n$ is *positive* when n is *even*. Therefore, when n is even, $a + b$ as well as $a - b$ is an exact divisor of $a^n - b^n$.

In $a^n + b^n$, n being even, put either $+b$ or $-b$ for a. Then $a^n + b^n$ becomes $b^n + b^n$, which is not zero. Therefore $a^n + b^n$ is never divisible by $a + b$ or $a - b$ when n is even.

In $a^n + b^n$, n being odd, put $-b$ for a. Then $a^n + b^n$ becomes $(-b)^n + b^n = 0$, since $(-b)^n$ is *negative* when n is *odd*. Therefore when n is odd $a + b$ is a divisor of $a^n + b^n$.

Summing up:

I. **$a^n - b^n$** is *always* divisible by **$a - b$**.

II. **$a^n - b^n$**, when n is *even*, is divisible both by **$a + b$** and by **$a - b$**.

III. **$a^n + b^n$** is *never* divisible by **$a - b$**.

IV. **$a^n + b^n$**, when n is *odd*, is divisible by **$a + b$**.

ORAL EXERCISES

State a binomial factor for each of the following:

1. $x^3 - y^3$.
2. $r^3 - 5^3$.
3. $27 - b^3$.
4. $m^5 - 2^5$.
5. $32x^5 - r^5$.
6. $a^8 - b^8$.
7. $a^{10} - b^{10}$.
8. $m^3 + 2^3$.
9. $a^3 + 8$.
10. $r^5 + y^5$.
11. $1 + r^7$.
12. $1 + x^5$.
13. $3^3 + 1$.
14. $4^3 - 1$.
15. $10^3 - 1$.

16. Is $10^5 + 1$ divisible by 11?

17. Is $10^9 - 1$ divisible by 9?

EXAMPLE

Factor $a^5 + b^5$.

Solution. By division,

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

Hence $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Note that the signs of the second factor are alternately plus and minus. Also note the order in which the exponents occur.

EXERCISES

Factor:

1. $x^5 + z^5$.
2. $r^5 + 1$.
3. $a^5 + 2^5$.
4. $x^5 + 32$.
5. $(a^2)^5 + (b^3)^5$.
6. $a^5 - b^5$.
7. $a^5 - 2^5$.
8. $a^5 - 32\,x^5$.
9. $a^7 - x^7$.
10. $1 - r^7$.
11. $x^7 - 128$.
12. $x^7 + 128\,a^7$.
13. $r^{10} + 32\,x^{10}$.
14. $x^7 + a^7$.
15. $1 + b^7$.
16. $x^5y^5 - 32$.
17. $r^7y^7 - 128$.
18. $a^{15} + 1$.
19. $1 - r^n$.

HINT. Write only the first five terms and the last term of the polynomial factor.

20. $1 + r^n$.
21. $x^6 - y^6$.
22. $x^8 - y^8$.
23. $a^{10} - b^{10}$.
24. $a^{12} - b^{12}$.
25. $(a + b)^5 - h^5$.

157. General directions for factoring. The following suggestions will prove helpful in factoring:

I. *First look for a common monomial factor, and if there is one (other than* 1), *separate the expression into its greatest monomial factor and the corresponding polynomial factor.*

II. *Then from the form of the polynomial factor determine with which of the following types it should be classed, and use the methods of factoring applicable to that type.*

1. $ax + ay + bx + by$.
2. $a^2 \pm 2\,ab + b^2$.
3. $a^2 - b^2$.
4. $x^2 + bx + c$.
5. $ax^2 + bx + c$.
6. $a^4 + ka^2b^2 + b^4$.
7. $a^3 \pm b^3$.
8. $a^n \pm b^n$.

III. *Proceed again as in II with each polynomial factor obtained, until the original expression has been separated into its prime factors.*

IV. *If the preceding steps fail, try the Factor Theorem.*

REVIEW EXERCISES

Factor:

1. $3r^2 + 6r^3 + 9r^4$.
2. $3\pi r^2 + 3\pi rh$.
3. $r^3 + 6r^2y + 9ry^2$.
4. $r^4 - 2a^2r^2$.
5. $a^2 + 4ab + 4b^2$.
6. $2c^3d - 8cd^3$.
7. $3m^3 - 3m^2 - 18m$.
8. $40a^3 - 5$.
9. $r^4 - 2br^3 + 4b^2r^2$.
10. $r^4 - 8a^3r$.
11. $(b + c)y + bz + cz$.
12. $r^3 - .04r$.
13. $2x^2 + 3ax + a^2$.
14. $x^4 - 7x^2y^2 + 9y^4$.
15. $a^4c - ac^4$.
16. $2x^7y - 2xy^7$.
17. $x^5 - 2x^4 - 9x^2$.
18. $\frac{3}{2}rx + \frac{3}{2}ry - \frac{3}{2}rz$.
19. $m^3 - 24m^2 + 144m$.
20. $ac + 2bc - ad - 2bd$.
21. $.04x^4 - .81y^2$.
22. $4b^5 + 5b^4c - 6b^3c^2$.
23. $x^3 + 4x - 5$.
24. $ax^2 - 4a + 3x^2 - 12$.
25. $x - x^2 - x^3 + x^4$.
26. $m^2 - n^2 - 2np - p^2$.
27. $b^2 - 2bc + c^2 - d^2$.
28. $x^3 - 8x^2 - x + 8$.
29. $a^8 + 27a^2$.
30. $64d^3 + 2d^8$.
31. $2x^3 - 2x^2 - 12x$.
32. $3b^7 - 3b$.
33. $1 - 5x^4 + 4x^8$.
34. $16m^4 - .0081$.
35. $18r^3 - 3r^2 - 36r$.
36. $(r^2 + s^2)^4 - 16r^4s^4$.
37. $(r^2 - 6)^2 - r^2$.
38. $16x^2y^2 - (x - y)^4$.
39. $a^4 + ab^{15}$.
40. $6x^2 - 13x + 6$.
41. $x^4 - x^2y^2 + x^3 - x^2y$.
42. $a^2 - 12a + 36 - a^4$.

43. $a^2 - b^2 + (a - b)^2$.
44. $m^4 - 7m^2n^2 + n^4$.
45. $32x^8 - x^3y^{10}$.
46. $5(a - b)^2 - a + b$.
47. $c^2 + 4d^2 - x^2 - 4cd$.
48. $x^2 - 20 + x^4$.
49. $6a^4 + 3a^3 - 3a^2$.
50. $16a^4 + 7a^2 + 1$.
51. $x^6y^6 - 64$.
52. $a^7 - a^2b^5 + a^2b - a^3$.
53. $x^2 - 5(2x - 5)$.
54. $4r^2 + 23r + 15$.
55. $a^4x^4 + 4$.
56. $x^3 - 10x - 3$.
57. $a^8 - 5a^4 + 4$.
58. $a^4 - 9a^2 - a + 3$.
59. $4ab + (a - b)^2$.
60. $x - 1 + x^5 - x^3$.
61. $x^3 + x - y - y^3$.
62. $2x^3 + 3x^2 + x$.
63. $m^7 - 8m - 7m^4$.
64. $2x^4 - 10x^2 + 4x$.
65. $10a - 7a^2 - 6a^3$.
66. $a^nx^2 + 2a^nx + a^n$.
67. $h^{2m} - 2h^mk^n + k^{2n}$.
68. $a^{2m} - b^{2n}$.
69. $a^{4m} + a^{2m}b^{2n} + b^{4n}$.
70. $a^{3m} + b^{3n}$.

158. Solution of equations by factoring. Review the principle, examples, and rule on pages 170–174.

EXERCISES

Solve by factoring:

1. $x^2 - 4 = 0$.
2. $x^2 - 5x = 0$.
3. $x^2 - 4a^2 = 0$.
4. $4r^3 = 25r$.
5. $a^3 = 64a$.
6. $r^2 + 5 = 6r$.
7. $2a^2 + a = 6$.
8. $x^2 - xs - 2s^2 = 0$.
9. $y^3 - y + 2 = 2y^2$.
10. $x^3 - ax^2 - 12a^2x = 0$.
11. $ar + 3r = a^2 - 9$.
12. $cx - c^2 + d^2 = dx$.
13. $5r^2 - 4r = 1$.
14. $2r^2 - 3r - 35 = 0$.

15. $24r^2 - 49r + 15 = 0.$
16. $x^3 + x^2 - a^2x - ax = 0.$
17. $y^3 - 7y - 6 = 0.$
18. $y^4 - 13y^2 + 36 = 0.$
19. $x^5 - 5x^3 = -4x.$
20. $x^4 - 7x^2 = -6x.$
21. $x^3 - x^2 - x + 1 = 0.$
22. $x^4 - 5x^2 + 4 = 0.$
23. $r^4 - 26r^2 = -25.$
24. $x^3 - x^2 - 14x + 24 = 0.$
25. $x^4 - 65x^2 + 64 = 0.$
26. $r^3 + 3r^2 + 3r + 1 = 0.$
27. $r^2 - 7r - 8 = r + 1.$
28. $4h^2 + h - 1 = 2h - 1.$

159. The highest common factor. *The highest common factor* (H. C. F.) of two or more monomials or polynomials is the expression of highest degree, with the greatest numeric coefficient, which is an exact divisor of each.

Thus the H. C. F. of $18r^2s^3$ and $27rs^4$ is $9rs^3$, and the H. C. F. of $x^2 - 4x + 3$ and $x^2 - 9$ is $x - 3$.

EXAMPLE

Find the H. C. F. of $4a^5 - 28a^4 + 48a^3$ and $8a^3 + 8a^2 - 96a$.

Solution. Factoring, we have

$$4a^5 - 28a^4 + 48a^3 = 4a^3(a-3)(a-4).$$
$$8a^3 + 8a^2 - 96a = 8a(a-3)(a+4).$$

Therefore, the H. C. F. is $2^2a(a-3)$ or $4a^2 - 12a$.

The method used in the preceding solutions for finding the H. C. F. of two or more monomials or polynomials is stated in the

Rule. *Separate each expression into its prime factors. Then find the product of such factors as occur in each expression, using each prime factor the least number of times it occurs in any one expression.*

If two or more polynomials have no common factor other than 1, then 1 is their H. C. F., and the polynomials are said to be prime to each other.

EXERCISES

Find the H. C. F. of the following:

1. 9, 12, 15.
2. 24, 60, 72.
3. 30, 45, 90.
4. $12\,a^2$, $30\,a^4$, $36\,a^5$.
5. $28\,a^2b^4c^3$, $42\,ab^5c$, $70\,a^3bc^2$.
6. $66\,ac^4x$, $132\,a^3c^2x^3$, $165\,a^2c^4x^2$.
7. $a^2 + 2\,ab + b^2$, $a^2 - b^2$.
8. $3\,a^2 - 3\,b^2$, $9(a-b)^2$, $3\,a^3 - 3\,b^3$.
9. $ax^2 - 2\,axy + ay^2$, $a^2x^2 - a^2y^2$, $2\,ax^3 - 2\,ay^3$.
10. $5\,x^7 - 160\,x^2$, $15\,x^5 - 60\,x^3$, $25\,x^7 - 200\,x^4$.
11. $4\,x^4y^4 - 4\,x^2y^6$, $5\,x^7y^4 - 5\,x^3y^8$, $8\,x^{11}y^4 - 8\,x^5y^{10}$.
12. $a^{2n} - 2\,a^nb + b^2$, $a^{2n} - b^2$, $a^{3n} - b^3$.

Note. The most famous, and in some respects the most perfect, treatise on elementary mathematics ever written is Euclid's "Elements." About one third of the material of the thirteen books treats topics which to-day would be considered arithmetic in character. In appearance and language, however, they are all geometric, for Euclid represents quantities not by numerals, as we do in arithmetic, or by letters, as we do in algebra, but by lines. Book VII contains the earliest statement of a general method for finding the Highest Common

Factor of two numbers. This method, though never necessary in elementary mathematical work, is of fundamental importance in advanced portions of algebra. It is so perfect and beautiful from a scientific point of view that until recently it remained in elementary treatises on algebra and arithmetic by force of tradition. It is a great tribute to Euclid's genius that he was able to devise so perfect a method for the process that all the efforts of two thousand years have been unable to improve it essentially.

FRACTIONS

160. Operations on fractions. Review the principle and explanation on pages 182–184, and the oral exercises on pages 183 and 185.

It should be noted that by the application of this principle a fraction is changed in form but not in value.

EXERCISES

Reduce to lowest terms:

1. $\frac{2a^2b^2}{10a^2b^3}$.

2. $\frac{169s^3t^4z^2}{65s^4t^3z^3}$.

3. $\frac{x^2-9}{2x^2+6x}$.

4. $\frac{5x^2+5xy}{25x^4+25x^2y^2}$.

5. $\frac{r^2-3r-4}{r^2+5r+4}$.

6. $\frac{s^2-1}{s^2+3s+2}$.

7. $\frac{2a^3+8a^2b+8ab^2}{5a^5-20a^3b^2}$.

8. $\frac{48x^4-6xy^3}{32x^6-32x^5y+8x^4y^2}$.

9. $\frac{a^3+6a^2b+9ab^2}{3a^2b+9ab^2}$.

10. $\frac{2c^2+2cd-4d^2}{5c^2-5d^2}$.

11. $\frac{r^3+r^2-5r-6}{r^3+8}$.

12. $\frac{a^3+b^3}{a^4+a^2b^2+b^4}$.

13. $\frac{r^5-s^5}{r^2-4rs+3s^2}$.

14. $\frac{m^3+n^3}{m^5+n^5}$.

15. $\dfrac{r^6 - 64}{r^6 + 4r^3 - 32}$.

16. $\dfrac{5x^4 - 40x}{3x^5 - 96}$.

17. $\dfrac{12a^3 + 10a^2 - 12a}{4 + 9a^2 - 12a}$.

18. $\dfrac{ab - ac}{ab + 3b - 3c - ac}$.

19. $\dfrac{3h^2 + 16h - 35}{5h^2 + 33h - 14}$.

20. $\dfrac{(\frac{2}{3})^2 - 1}{\frac{2}{3} + 1}$.

21. $\dfrac{(\frac{3}{5})^3 - 1}{\frac{3}{5} - 1}$.

22. $\dfrac{(\frac{3}{2})^2 - (\frac{3}{2})(\frac{2}{5}) + (\frac{2}{5})^2}{(\frac{3}{2})^3 + (\frac{2}{5})^3}$.

23. $\dfrac{1 - r^2}{(1 + rs)^2 - (r + s)^2}$.

24. $\dfrac{r^2 + r + s - s^2}{1 - (r - s)^2}$.

161. Changes of sign in a fraction. Review the explanation and principle on pages 198–199.

ORAL EXERCISES

Express the following as equivalent fractions in three additional ways, using the principle on page 199:

1. $\dfrac{-a}{b}$.

2. $\dfrac{r}{2 - t}$.

3. $-\dfrac{2n}{1 - n}$.

Without changing their values, change the signs in the following indicated products:

4. $(s - 2)(s + 2)$.

5. $(x - a)(x - b)$.

6. $-(x + y)(x - y)$.

7. $-x(a - b)(c - d)$.

8. $(x - a)(x - b)(c - x)$.

9. $(x + 1)(x - 1)(x^2 + 1)$.

10. Find the indicated product in Exercise 8. Then change the sign of two factors and again find the product. Compare the results.

11. Make a general statement of which the result of Exercise 10 is an illustration.

State equivalent fractions for the following, each of which shall contain the factor $x - 1$ in its denominator:

12. $\frac{1}{1 - x}$. 13. $-\frac{2}{1 - x^2}$. 14. $\frac{s - 2}{x(1 - x)^3}$.

15. $\frac{-3d - 5}{-1 + 2x - x^2}$. 16. $-\frac{t^2 - t + 1}{1 - x^3}$.

By changes of sign make the denominators of the following fractions alike, without changing the values of the fractions:

17. $\frac{3}{(a - 2)(a + 2)}$, $\frac{a + 3}{(2 - a)(2 + a)}$.

162. Lowest common multiple. Review the examples and rules on pages 187–188.

EXERCISES

Find the L. C. M. of the following:

1. $a^3 + ab^2$, $a^2 - 2ab + b^2$, $a^2 - ab$.
2. $ax - ay + bx - by$, $a^2 - b^2$, $x^2 - y^2$.
3. $c^2 - 6cd + 9d^2$, $c^2 - 9d^2$, $c^2 + 3cd$.
4. $t^2 - tr - 20r^2$, $t^2 - 25r^2$, $t^2 - 16r^2$.
5. $r^2 - r - 2$, $r^2 - 8r + 12$, $r^2 - 5r - 6$.
6. $k^2 - k - 6$, $k^2 - 6k + 9$, $6k - 18$.
7. $n^5 - 32$, $4 - n^2$, $5n^2 + 10n$, $5n - 10$.
8. $m^3 - 8$, $4 - m^2$, $m^3 + 4mr^2 + 2m^2r$.
9. $a^3 - 7a + 6$, $a^2 - 3a + 2$, $a^2 + 2a - 3$.

163. Equivalent fractions. Review the examples and rules on pages 189–190.

EXERCISES

Change to respectively equivalent fractions having the lowest common denominator:

1. $\frac{2}{5x}, \frac{3}{2x^2}$.

2. $\frac{x-2}{6x^3y^4}, \frac{2}{3x^5y}$.

3. $\frac{2x}{x-2}, \frac{3}{x+2}$.

4. $\frac{3}{x^2-9}, \frac{2x}{x-3}$.

5. $\frac{3a}{(a-2)^2}, \frac{2a^2}{a-2}$.

6. $\frac{3x+y}{x-y}, \frac{1}{4(x-y)^2}, xy$.

7. $\frac{2a^2}{2a+4}, \frac{a+3}{3a+1}, \frac{2a^2+5}{6a^2+14a+4}$.

8. $\frac{2x+1}{x^2-2x+4}, \frac{5}{x^3+8}, \frac{1}{x+2}$.

9. $\frac{ax-bx-ar+br}{ax+bx+ar+br}, \frac{x^2-r^2}{a^2-b^2}, \frac{a^2-b^2}{x^2-r^2}$.

10. $\frac{2a}{8-a^3}, \frac{5a}{a-2}, \frac{4}{a^2-4}, \frac{3a+2}{a^3+2a^2+4a}$.

164. Addition and subtraction of fractions. To find the algebraic sum of two or more fractions in their lowest terms we proceed as in the examples on pages 193, 195, and 197.

EXERCISES

Find the algebraic sum of:

1. $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{1}{16}$.

2. $\frac{2a}{5}+\frac{3a}{10}-\frac{2a}{15}$.

3. $\frac{3}{r}+\frac{5}{2r}-\frac{11}{3r}$.

4. $\frac{2c}{3a}-\frac{5c^2}{6a^2}+\frac{c}{9a}$.

5. $\frac{4c}{3a^2}-\frac{5c^2}{2a^2}+\frac{6c}{5a^3}$.

6. $\frac{5a}{a-3}-\frac{2}{3}$.

7. $\frac{7b}{2x} - \frac{b}{10x} + \frac{3b-1}{5^2x^2}.$

8. $\frac{s}{s-1} - \frac{2}{3} - \frac{5s-2}{s^2-1}.$

9. $2 + \frac{5r}{r-7}.$

HINT. $2 + \frac{5r}{r-7} = \frac{2}{1} + \frac{5r}{r-7}.$

10. $x + 3 - \frac{x^2}{x-3}.$

11. $a - \frac{a-6}{2a-3} - 2.$

HINT. $a - \frac{a-6}{2a-3} - 2 = a - 2 - \frac{a-6}{2a-3}.$

12. $\frac{7}{x^2-49} - \frac{3}{x^2-6x-7}.$

13. $\frac{r-y}{3r+2y} + \frac{12ry}{9r^2-4y^2} - \frac{r+y}{3r-2y}.$

14. $\frac{k-3}{k^2-2k-3} - \frac{2-k}{k^2-3k+2} - \frac{1}{1-k^2}.$

15. $\frac{b^2-16b}{4-b^2} - \frac{3+2b}{b-2} + \frac{3b-2}{b+2}.$

16. $\frac{s}{1+s} - \frac{1-s}{s} + \frac{2s}{s^2-1} - \frac{1}{1-s}.$

17. $\frac{a+2}{a^2-7a+12} - \frac{a+2}{6+a^2-5a} - \frac{1}{6a-a^2-8}.$

18. $\frac{a+b}{(b-c)(c-a)} + \frac{b+c}{ac-a^2-bc+ab} + \frac{a+c}{(a-b)(b-c)}.$

165. Multiplication and division of fractions. The method of multiplying fractions is illustrated in the examples on pages 204–205. The method to be followed in the division of fractions is given in the rule on page 207 and the note on page 208.

EXERCISES

Perform the indicated operations:

1. $\frac{3a}{25x^2m}\cdot\frac{10xm^2}{2a^2x}$.

2. $2c\cdot\frac{3b}{5cb^2}\cdot\frac{4c}{9b}$.

3. $\frac{8r^2s}{t^4}\cdot\frac{3s^2}{6xs^2}\cdot\frac{20r^2x^2}{5r^3t^3}$.

4. $4\frac{1}{2}\cdot\frac{2m^2}{3t^3}\cdot\frac{t^2}{15m}\cdot\frac{5tm}{mt^2}$.

5. $20a^2\cdot\frac{14x}{4a^6}\cdot\frac{a^2}{42x^4}$.

6. $\frac{n^2-9}{2n^2}\cdot\frac{4n}{2n-6}$.

7. $\frac{2x^2+6}{5z^3}\cdot\frac{10z^4}{3x^2+9}$.

8. $\frac{r^2-4}{y-z}\cdot\frac{3y-3z}{r^2+6r+8}$.

9. $\left(3a+\frac{1}{a}\right)\left(\frac{a^2}{9a^4-1}\right)\left(a-\frac{1}{3a}\right)$.

10. $\frac{2r^2-13r+15}{4r^2-9}\cdot\frac{2r+1}{2r-1}\cdot\frac{2r-1}{r-5}$.

11. $\frac{s^2-11s+30}{s^3-6s^2+9s}\cdot\frac{s^2-3s}{s^2-25}\cdot\frac{s^2+2s-15}{s^2-9s}$.

12. $\frac{24r^2}{99ry^2}\cdot\frac{8y^2z}{22r^2y}\div\frac{48r^3z}{121r^2y^3}$.

HINT. See Rule, Section 14.

13. $\frac{5x^2}{x-y}\cdot\frac{(x-y)^2}{20x^2}\div\frac{1}{x(x-y)}$.

14. $\frac{r^2-rx-6x^2}{r^3-9rx^2}\div\frac{r+2x}{r+3x}$.

15. $\frac{x^2-8x+15}{6-5x+x^2}\div\frac{7x-10-x^2}{4-4x+x^2}$.

16. $\frac{r^4+64s^4}{r^2-s^2}\cdot\frac{r+s}{r^2-4rs+8s^2}\div\frac{r^2+4rs+8s^2}{r^2-2rs+s^2}$.

Simplify each term in Exercises 17–18:

17. $\left(\frac{3r^2}{2s}\right)^4 - 4\left(\frac{3r^2}{2s}\right)^3\left(\frac{2s^2}{3r}\right) + 6\left(\frac{3r^2}{2s}\right)^2\left(\frac{2s^2}{3r}\right)^2.$

18. $\left(\frac{2a}{5x^2}\right)^3 - 3\left(\frac{2a}{5x^2}\right)^2\left(\frac{5x^3}{4a^2}\right) + 3\left(\frac{2a}{5x^2}\right)\left(\frac{5x^3}{4a^2}\right)^2.$

Perform the indicated operations:

19. $\left(\frac{s}{3} - \frac{3}{s}\right) \div \left(\frac{s^2 - 6s + 9}{3s}\right).$

20. $\frac{a}{a^2 - 4} \div \left(\frac{a}{3a^2 + 7a + 2}\right)\left(3a - \frac{2}{a} - 5\right).$

21. $$\frac{x + 1 + \frac{4}{x + 3}}{x - 5 + \frac{12}{x + 3}}.$$

HINT. An expression of this form is called a complex fraction. It is simply another way of writing

$$\left(x + 1 + \frac{4}{x + 3}\right) \div \left(x - 5 + \frac{12}{x + 3}\right).$$

22. $$\frac{2 + \frac{x + 6}{x - 2}}{3 + \frac{5}{x - 2}}.$$

24. $$\frac{\left(a - \frac{1}{a^2}\right)^2\left(\frac{2}{a - 1}\right)}{\left(a + 1 + \frac{1}{a}\right)}.$$

23. $$\frac{x - 1 - \frac{3}{x}}{x + 1 + \frac{x}{x - 3}}.$$

25. $$\frac{\frac{2}{b} - \frac{1}{a + b} + \frac{1}{a - b}}{\frac{a + b}{a - b} - \frac{a - b}{a + b}}.$$

26. $$\frac{a^2 + b^2}{a^2 - b^2} - \frac{\frac{b}{a + b}}{1 + \frac{2b - a}{a - b}}.$$

27. $1-\dfrac{1}{1-\dfrac{1}{1+\frac{1}{2}}}$.

HINT. First simplify $1+\frac{1}{2}$. Then $1-\frac{1}{\frac{3}{2}}$, etc.

28. $1-\dfrac{1}{1+\dfrac{2}{1-\frac{2}{3}}}$.

29. $$\frac{\left(1+\dfrac{x}{a+b}+\dfrac{x^2}{(a+b)^2}\right)\cdot\left(1-\dfrac{x^2}{(a+b)^2}\right)}{\left(1-\dfrac{x^3}{(a+b)^3}\right)\left(1+\dfrac{x}{a+b}\right)}.$$

166. Equations involving fractions. Review the explanations, examples, and rule on pages 215–217, 222, and 224.

EXERCISES

Solve and check:

1. $\dfrac{7x}{3}-1=\dfrac{5x}{3}+3.$

2. $\dfrac{11x}{9}-\dfrac{2}{3}=x.$

3. $\dfrac{3r}{2}-\dfrac{16}{3}=r-\dfrac{25}{6}.$

4. $\dfrac{3r+14}{3}-\dfrac{6-r}{4}=-1.$

5. $\dfrac{2(k-4)}{3}-1=\dfrac{k+5}{3}.$

6. $\dfrac{3}{4x}+\dfrac{7}{16}=\dfrac{4}{3x}.$

7. $\dfrac{8}{7x+3}=\dfrac{3}{3x+1}.$

8. $\dfrac{3x-3}{x+7}+\dfrac{3}{5}=0.$

9. $(x+5)(x+1)-(x-3)(x-2)=10.$

10. $\dfrac{3r-5}{5r-5}+\dfrac{5r-1}{7r-7}-\dfrac{r-4}{1-r}=2.$

11. $\frac{x+a}{a} - \frac{x+b}{b} = b - a.$

12. $\frac{x}{a} - a = \frac{x}{b} - b.$

13. $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$

14. $\frac{x}{a+2} - \frac{x+16}{a-2} = 4a.$

15. $\frac{2x+1}{x+3} + \frac{3x-7}{2-x} = \frac{9-3x-x^2}{x^2+x-6}.$

16. $\frac{x+cd}{c+d} - \frac{x-cd}{c-d} = \frac{2c^2d-2cd^2}{c^2-d^2}.$

17. $\frac{x-a}{2x-a} - \frac{3x-c}{6x-c} = 0.$

18. $\frac{3s+4a}{s+2a} + \frac{3s-5a}{4a-s} = \frac{10a^2}{s^2-2as-8a^2}.$

19. $\frac{5.3r-3.7}{13} = \frac{12.5r-8.06}{14}.$

20. $12.2 - \frac{5-2x}{.5} = 2.3 - (5+7x) + \frac{2-4x}{2}.$

21. $\frac{3x-1}{.25} + \frac{x-4}{.5} = 3(3x-14).$

22. $\frac{x-\frac{1}{3}}{x+\frac{2}{3}} = \frac{x+1}{x+\frac{4}{3}}.$

PROBLEMS

1. Separate 175 into two parts such that one part will be $\frac{2}{5}$ of the other.

2. Separate 360 into two parts such that the greater will be $3\frac{1}{11}$ times the less.

3. Separate 336 into two parts such that $\frac{1}{3}$ of the greater shall equal $\frac{3}{5}$ of the less.

4. By what number must 875 be divided so as to give the partial quotient 32 and a remainder 11?

5. Separate 175 into two parts such that their quotient is $5\frac{1}{4}$.

6. One third of a pile is in earth, one fifth is in water, and 14 feet are above water. How long is the pile?

7. A certain pile is 57 feet long. The part in earth is $\frac{2}{3}$ the part in water; the part in water is $\frac{2}{3}$ the part in air. How many feet of it are in water? in earth? in air?

8. A boy is 14 years old and his sister is 9 years old. In how many years will the boy be $\frac{7}{6}$ as old as his sister?

9. The square of a certain number is 785 greater than $\frac{8}{9}$ the product of the next two consecutive numbers. Find the number.

10. The denominator of a fraction is 4 times the numerator. If the numerator is increased by 4 and the denominator is decreased by 4, the resulting fraction is $\frac{3}{2}$. What are the terms of the fraction?

11. A man sold $\frac{1}{3}$ of his salt and $\frac{2}{3}$ pound more. He then sold $\frac{1}{5}$ of what remained and $\frac{4}{5}$ pound more; he then sold $\frac{1}{3}$ of the rest and had 8 pounds left. How much had he at first?

12. A rectangle is $3\frac{1}{3}$ times as long as it is wide. If it were 8 yards shorter and $2\frac{1}{2}$ yards wider, its area would be 108 square feet less. Find the dimensions of the rectangle.

13. A square court has $\frac{9}{8}$ the area of a rectangular court whose length is 5 yards greater and whose width is 2 yards less than the side of the square. What are the dimensions of the square court?

14. A, B, and C together have \$5460. A has \$50 more than $\frac{2}{3}$ as much as B, and C has \$45 more than $\frac{3}{4}$ as much as B. How much has each?

15. If a man can do in 3 days a piece of work which takes a boy 7 days, how long will it take both working together to complete the work?

16. A can do a piece of work in 8 days and B in 10 days. How many days will they both require working together?

17. A can do a piece of work in 10 days, and B in 15 days. After they have worked together 5 days, how long will it take A to finish it alone? B alone?

18. A can do a piece of work in 6 days, B in 8 days, and C in 9 days. A works 2 days and stops; B works 2 days, when C joins him. How long must C work before the job is completed?

19. A tank has a supply pipe that fills it in 5 hours and a waste pipe that empties it in 7 hours. If the tank is empty and both pipes are open, how much time must elapse before the tank is filled?

20. One pipe can fill a swimming tank in 27 minutes and another pipe can fill it in 30 minutes. A waste pipe can empty the tank in 20 minutes. With all pipes open, how long will it be before the tank is filled?

21. The sum of the numerator and denominator of a certain fraction is 36. If 2 be added to the numerator and subtracted from the denominator, the resulting fraction is $\frac{1}{3}$. What was the original fraction?

22. The length of a rectangle is $\frac{2}{3}$ its width. The perimeter is 150 feet. What are the dimensions of the rectangle?

23. The diameter of the earth is $3\frac{2}{3}$ times that of the moon, and the difference of the two diameters is 5760 miles. Find each diameter in miles.

24. The diameter of the sun is 3220 miles greater than 109 times the diameter of the earth, and the sum of the diameters is 874,420 miles. Find each diameter in miles.

25. A man rowed 12 miles up a river in 5 hours and back in 3 hours. What is his rate in still water?

26. A man who can row 4 miles per hour in still water finds that it requires $5\frac{1}{2}$ hours to row upstream a distance which it requires $2\frac{1}{2}$ hours to row down. Find the speed of the current.

27. A passenger train whose speed is 45 miles per hour leaves a station 3 hours and 45 minutes after a freight train. The passenger train overtakes the freight in 4 hours and 15 minutes. Find the speed of the freight train in miles per hour.

28. A, B, and C can do a piece of work in 12 days, B and C together in 20 days, and C alone in 45 days. How long does it take A alone to do the work? B alone?

29. A man invests a part of $9000 at 5% and the remainder at $4\frac{1}{2}$%. If the yearly interest on the whole investment is $423, how much was invested at each rate?

30. A man invests $6800 in two parts: the first part at $5\frac{1}{2}$%, the second at $7\frac{1}{5}$%. If the average rate of interest is $6\frac{9}{20}$%, find the amount of each investment.

31. How much water must be added to a gallon of alcohol 90% pure so as to make a mixture 10% pure?

HINT. Let w = the number of gallons of water to be added.

Then
$$\frac{\frac{90}{100}\cdot 1}{1+w}=\frac{10}{100}.$$

32. How much water must be added to 25 gallons of milk containing 8% butter fat to make a mixture containing 5% butter fat?

33. It is desired to mix coffee which sells for 35¢ per pound with coffee which sells for 50¢ a pound so as to obtain a ten-pound mixture which may be sold for 45¢ a pound. How many pounds of each kind of coffee must be used?

34. How many quarts of 45¢ vinegar must be mixed with 10 quarts of 25¢ vinegar so that a quart of the mixture may sell for 40¢?

35. A certain lot of pig iron contains 91% iron. How much pure iron must be melted with 10 tons of pig iron to make iron 97% pure?

36. The arms of a balanced lever are 8 feet and 13 feet respectively. The shorter arm carries 39 pounds. Find the load on the longer arm.

37. If the load on the longer arm in Problem 36 be reduced 8 pounds, how many feet from the fulcrum must a 26-pound weight be placed on the shorter arm to restore the balance?

38. How many ounces of water must be added to 1 ounce of carbolic acid which is 90% pure to make a 3% solution?

39. At what time between 4 and 5 o'clock will the hands of a clock be together?

Solution. The minute hand moves twelve times as fast as the hour hand. While the minute hand travels x spaces, the hour hand travels $\frac{x}{12}$ spaces. Hence, $x - \frac{x}{12}$ equals the number of spaces gained by the minute hand in any given time x.

In the time from 4 o'clock until the hands are together, the minute hand must gain 20 minute spaces on the hour hand to overtake it.

Therefore $$x - \frac{x}{12} = 20.$$

Whence $$x = 21\tfrac{9}{11}.$$

Hence, the hands are together $21\tfrac{9}{11}$ minutes after 4 o'clock.

40. At what time between 9 and 10 o'clock are the hands of a clock together?

41. At what time between 4 and 5 o'clock are the hands of a clock in a straight line?

42. At what time between 7 and 8 o'clock will the minute hand be 10 minute spaces ahead of the hour hand? Ten minute spaces behind?

LINEAR SYSTEMS

167. Graphical solution of a linear system in two unknowns. Review the construction of the graph of a single linear equation in two unknowns on page 276. The *graphical solution* of a system of linear equations in two unknowns is explained and illustrated on pages 283–284.

EXERCISES

Solve graphically and check:

1. $2x + y = 8,$
$x + 2y = 13.$

2. $x - y = 6,$
$3x + 4y = -17.$

3. $4x - 6y = 0,$
$5y - 3x = -1.$

4. $5x - y = 8,$
$x - 5y = 16.$

5. $x + y = 5,$
$y + 2 = 0.$

6. $x + 5 = -3y,$
$6y + 2x - 11 = 0.$

7. $x + y = 4,$
$x + 2y = 7.$

8. $x - y = 5,$
$3x + 2y = 5.$

168. Elimination. In order to find values of x and y which satisfy the equation

$$2x + 5y = 15 \tag{1}$$

when we know that

$$y = x - 4 \tag{2}$$

we may substitute for y in the first equation the value of y from the second, obtaining the single equation in x,

$$2x + 5(x - 4) = 15, \text{ or } 7x = 35, \text{ or } x = 5.$$

The process by which we have obtained one equation containing one unknown from the two equations (1) and (2), each of which contains two unknowns, illustrates one method of elimination.

In general, the process of deriving from a system of n equations a system of $n - 1$ equations, containing one variable less than the original system, is called **elimination**, since one variable is eliminated.

For example, if we have a system of two equations in two unknowns, the process of elimination leads to one equation in one unknown. Since we can always solve such an equation, it appears that we can solve a system of two equations in two unknowns whenever it is possible to eliminate one of the unknowns. We shall see that only in certain exceptional cases is elimination impossible. This is either because more than one unknown is removed when we try it or because the result of attempted elimination is not an equation.

Only two methods of solution will be considered — that involving **elimination by substitution** and that involving **elimination by addition or subtraction.**

169. Solution by substitution. Review pages 258–259 for the solution by substitution of a system of two linear equations.

EXERCISES

Solve by substitution:

1. $x + y = -1,$
$5x - y = 13.$

2. $2x - 3y = 2,$
$5y - 4x = -6.$

3. $2x + y = 8,$
$x - 2y = 9.$

4. $2(x + y) + 3y = 4,$
$5 = x + y.$

5. $4x + 7 = 3y,$
$x + y = 0.$

6. $6x + 38 = 12y,$
$4x - 8y = 0.$

7. $4x - y = -2,$
$5x + y = 6\frac{1}{2}.$

8. $5x + \frac{3}{2}y = 7\frac{5}{6},$
$x - 4y = -11\frac{1}{3}.$

170. Solution by addition or subtraction. Review pages 254–257 for the solution by addition or subtraction of a system of two linear equations.

171. Special cases. A system of equations which has a common set of roots is called a **simultaneous** system. The equation $x + y = 8$ has as roots any set of two numbers whose sum is 8. If $x + y = 4$ is taken as the other equation of a system, one can see immediately that the two equations have no set of roots in common, since the sum of two numbers cannot be 8 and 4 at the same time.

Graphs of these two equations give parallel lines. Since two parallel lines cannot intersect, we see clearly why no set of roots exists for $x + y = 8$ and $x + y = 4$.

A system of equations which does not have a common set of roots is called an **inconsistent** or **incompatible** system.

The attempt to solve an incompatible system results in getting rid not only of one but of both unknowns and leads to a statement in the form of an equation which is false.

Consider $$x + y = 12, \qquad (1)$$

$$x + y = 6, \qquad (2)$$

(1) – (2), $$0 = 6, \text{ which is false.}$$

If, on the other hand, the equation $x + y = 12$ is taken for one equation of a system and $2x + 2y = 24$ for the other, it appears that any set of numbers which satisfies one equation also satisfies the other, since if the sum of two numbers is 12, the sum of twice those numbers is 24, and any one of the countless sets of roots of one equation is a set of roots of the other. In fact, the second equation may be obtained from the first by multiplying each member by 2.

If one equation of a system can be obtained from one or more of the other equations of the system by application of one or more of the axioms, it is called a **derived** or **dependent** equation. If it cannot be so obtained, it is called **independent**.

Thus equations (1) and (2) in the example on page 258 are independent, while equation (5) is derived from them.

The graphs of $x + y = 12$ and $2x + 2y = 24$ are identical lines. Hence any point on the first line is a point on the second.

EXERCISES

Solve the following systems by addition and subtraction:

1. $2x - y = 9,$
$x + 7y = 12.$

2. $3x + 4y = -6,$
$5x - 3y = 19.$

3. $4x + 5y = -25,$
$7x - 6y = -29.$

4. $3x - 4y = -4,$
$x + y = 2\frac{1}{6}.$

5. $3x + 5y = 5,$
$9x - 6y = -27.$

6. $2x + 4y = -22,$
$5x + y = 35.$

7. $7x + 10y = -50,$
$5y - 3x = 27.$

8. $4x + 3y = 4.1,$
$5x - 7y = -2.4.$

9. $\frac{r}{5} = \frac{s}{4},$
$3r + \frac{1}{2}s = 17.$

10. $\frac{3y+1}{4} - \frac{z+22}{12} = 3,$
$z - 2y = 1.$

11. $.5x + .7y = 9\frac{3}{10},$
$.8x - .2y = 3.$

12. $\frac{r+1}{3} = \frac{2-s}{2} - \frac{5}{6},$
$\frac{3r-2}{3} = \frac{s-3}{4} - \frac{5}{12}.$

13. $\frac{1}{x} + \frac{1}{y} = -\frac{1}{6},$
$\frac{2}{x} - \frac{3}{y} = \frac{4}{3}.$

HINT. Solve first for $\frac{1}{x}$ and $\frac{1}{y}$.

14. $\frac{3}{x-3} - \frac{1}{y} = 0,$
$\frac{x-6y}{5} = \frac{2}{5}.$

15. $\frac{3x+z}{z-x} + 2 = \frac{12}{x-z},$
$\frac{x+7}{z} + \frac{17+x}{5-z} = 0.$

Solve for x and y:

16. $2x - 3y = 9b,$
$5x + 9y = 6b.$

17. $5x + 4y = 10a + 4,$
$x - 2ay = 0.$

18. $2ax - 3by = 7,$
$5ax + 7by = 3.$

19. $\dfrac{a}{x} + \dfrac{3a}{y} = 1,$
$\dfrac{3a}{x} + \dfrac{a}{y} = \dfrac{1}{2}.$

20. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a+b}{ab},$
$x - y = \dfrac{a^2 - b^2}{ab}.$

21. $2ax - 4by = 6c,$
$3ax - 5by = 5c.$

22. $ax + by = c,$
$kax + kby = ck.$

23. $ax + by = c,$
$ax + fy = g.$

172. Equations in several unknowns. We have already seen that the equation $x + y = 8$ is satisfied by an unlimited number of sets of roots, since there is an infinity of pairs of numbers whose sum is 8.

An equation or a system of equations which is satisfied by an infinite number of sets of roots is said to be **indeterminate.**

If a simultaneous system is satisfied by only a limited number of sets of roots, it is said to be **determinate.**

The system $x + y = 8$, $x - y = 4$ is determinate and has the set of roots (6, 2). The system $2x + 2y = 16$, $x + y = 8$, is indeterminate.

When the number of unknowns in a system of linear equations exceeds the number of equations, the system is indeterminate. If the number of equations equals the number of unknowns, the system is usually determinate and simultaneous. If the number of equations exceeds the number of unknowns, the system is usually inconsistent. There are many special cases which arise in the study of linear systems in n unknowns in section 173, but they become very complicated for larger values of n, and a thorough study of them is quite beyond the scope of this text.

NOTE. It is not a little remarkable that the writings of the first great algebraist, Diophantos of Alexandria (about A.D. 275), are devoted almost entirely to the solution of indeterminate equations; that is, to finding the sets of related values which satisfy an equation in two unknowns, or perhaps, two equations in three unknowns. We know practically nothing of Diophantos himself, except the information contained in his epitaph, which reads as follows: "Diophantos passed one sixth of his life in childhood, one twelfth in youth, one seventh more as a bachelor; five years after his marriage a son was born who died four years before his father, at half his father's age." From this statement the reader was supposed to be able to find at what age Diophantos died. As a mathematician Diophantos stood alone, without any prominent forerunner or disciple, so far as we know. His solutions of the indeterminate equations were exceedingly skillful, but his methods were so obscure that his work had comparatively little influence upon later mathematicians.

173. Determinate systems. The method of obtaining the set of roots of a determinate system in three unknowns is illustrated in the

EXAMPLE

$$\text{Solve the system}\begin{cases} x + 4y - 3z = 21, & (1)\\ 2x - 5y + z = -6, & (2)\\ 5x - 3y + 2z = 3. & (3)\end{cases}$$

Solution. First eliminate one unknown, say z, between (1) and (2):

$$x + 4y - 3z = 21. \qquad (1)$$

$$(2)\cdot 3, \qquad 6x - 15y + 3z = -18. \qquad (4)$$

$$(1)+(4), \qquad 7x - 11y = 3. \qquad (5)$$

Now eliminate z between (2) and (3):

$$(2)\cdot 2, \qquad 4x - 10y + 2z = -12. \qquad (6)$$

$$5x - 3y + 2z = -3. \qquad (3)$$

$$(6)-(3), \qquad -x - 7y = -9. \qquad (7)$$

The equations (5) and (7) contain the same two unknowns x and y.

$$7x - 11y = 3. \qquad (5)$$

(7) · 7 $$-7x - 49y = -63. \qquad (8)$$

(5) + (8), $$-60y = -60.$$

$$y = 1.$$

Substituting in (7), $$x = 2.$$

Substituting both these values in (1),

$$2 + 4 - 3z = 21.$$

Whence $$z = -5.$$

Check. Substituting 2 for x, 1 for y, and -5 for z in (1), (2), and (3) respectively,

$$2 + 4 \cdot 1 - 3(-5) = 21, \text{ or } 21 = 21.$$
$$2 \cdot 2 - 5 \cdot 1 + (-5) = -6, \text{ or } -6 = -6.$$
$$5 \cdot 2 - 3 \cdot 1 + 2(-5) = -3, \text{ or } -3 = -3.$$

For the solution of a simultaneous system of equations in three unknowns we have the

RULE. *From an inspection of the coefficients decide which unknown is most easily eliminated.*

Using any two equations, eliminate that unknown.

With one of the equations just used and the third equation again eliminate the same unknown.

The last two operations give two equations in the same two unknowns. Solve these equations.

Substitute in the simplest of the original equations the two values found, and solve for the third unknown.

CHECK. *Substitute the values found in the original equations and simplify results.*

A system of four independent equations each of which contains the same four unknowns may be solved as follows:

Use the first and second equation, then the first and third, and lastly the first and fourth, and eliminate the same unknown each time. This gives a system of three equations in the same three unknowns, which can be solved by the rule given above.

EXERCISES

Solve for x, y, z, and w, and check the results:

1. $x + 3y - 5z = 2,$
$2x - y - z = 1,$
$3x + 5y - 7z = -10.$

2. $2x + 3y + 4z = -14,$
$x - y + 3z = 0,$
$5x + 2y + z = 14.$

3. $2x + y - 3z = 5,$
$x + 2y + z = 11,$
$3x - y + 2z = 4.$

4. $x + 2y + z = -1,$
$2x - y + z = -20,$
$-x - y - 5z = 13.$

5. $2x + 3y - 3z = -2,$
$4x - 2y + z = 9,$
$3x + y + 2z = 13.$

6. $2x + 3y + z = 14,$
$x - y + z = 4,$
$x + y - z = 0.$

7. $x + y + z = 1,$
$x + y - z = 2,$
$x - y + z = 3.$

8. $x + 2y + z = 1,$
$2x + y - z = 0,$
$x + 2y + z = 0.$

9. $4x - 3y = z,$
$z = x + y,$
$2x = 3y + 1.$

10. $x - 2y = 10,$
$3y + 1 = -4z,$
$5x - 18 = z.$

11. $x = 3z + 2,$
$y = x - 7\frac{1}{2},$
$z = 6y - 1.$

12. $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3,$
$\frac{a}{x} - \frac{b}{y} + \frac{2c}{z} = 2,$
$\frac{5a}{x} - \frac{2c}{z} = \frac{3b}{y}.$

13. $ax + by = 1,$
$by + cz = 1,$
$cz + ax = 1.$

14.
$$\begin{aligned} cy + bz &= bc, \\ az + cx &= ca, \\ bx + cy &= ab. \end{aligned}$$

15.
$$\begin{aligned} 2x + y + z + w &= 1, \\ x - y - z + 2w &= 4, \\ x + 2y - z - w &= 0, \\ x - y + 2z - w &= 1. \end{aligned}$$

16.
$$\begin{aligned} 2x - y + z - w &= 2, \\ 3x + y - z - w &= 0, \\ 4x - 2y + z - w &= -9, \\ 2x - 3y - 2z + w &= 7. \end{aligned}$$

17.
$$\begin{aligned} x + y + z &= 6, \\ x + y + w &= 7, \\ x + z + w &= 8, \\ y + z + w &= 9. \end{aligned}$$

PROBLEMS

1. The sum of two numbers is 139 and their difference is 39. Find the numbers.

2. The sum of two numbers is 17. Twice one less the other equals 7. What are the numbers?

3. The sum of two numbers is 13. Five times one less the other equals 29. What are the numbers?

4. There are three numbers such that the sum of the first and second is 67, the sum of the first and third is 79, and the sum of the second and third is 88. What are the numbers?

5. Forty-four tons of sand and gravel are required for the foundation of a building. It is found that the material can be hauled in a given time either by 12 trucks and 4 drays, or by 8 trucks and 10 drays. What is the capacity of a truck and of a dray?

6. What temperature on the Centigrade scale is one half that on the Fahrenheit scale?

HINT. Let the second equation be $C = \frac{1}{2} F$.

7. What temperature on the Fahrenheit scale is 3 times the equivalent temperature on the Centigrade scale?

8. Two men travel from San Francisco to Los Angeles by rail. It costs one of them three times as much for excess baggage as it costs the other. One pays $19.10 in all, the other $22.30. How much does each pay for his ticket?

9. If $cx + dy = 2$ is satisfied by $x = 2$ and $y = 3$, and also by $x = 3$ and $y = 4$, what values must c and d have?

10. If $2x + ay = m$ is satisfied when $x = 1$ and $y = -1$ and also when $x = 5$ and $y = 4$, what values must a and m have?

11. A boy rows 8 miles with the current in 1 hour 4 minutes, and returns against the current in $2\frac{2}{7}$ hours. At what rate would he row in still water? What is the rate of the current?

12. A and B can do a piece of work in s days, A and C in t days, and B and C can do it in r days. In how many days can each do the work?

13. A has a certain capital which is invested at a certain interest. B has $5000 more capital than A, and invests it at 1 per cent more. His yearly income exceeds that of A by $400. C has $7500 more capital than A and invests it at 2 per cent more. His income exceeds that of A by $750. Find the capital of each and the rate at which it is invested.

14. A bag weighing 23 ounces contains two sizes of steel balls — ounce balls and $\frac{3}{5}$ ounce balls. There are 31 balls in all. Find the number of balls of each size.

15. A man has $3.55 in nickels and quarters. If he has 39 coins in all, how many has he of each?

16. A collection of 53 coins consists of nickels, dimes, and quarters. The value of the coins is $8.40. Three times the number of quarters less twice the number of dimes is 7 less than 4 times the number of nickels. How many of each kind are there?

17. A man has $12.00 in quarters, dimes, and nickels. He has as many quarters as he has dimes, and three times as many nickels as dimes. How many of each has he?

18. The difference between the second and third angles of a triangle is 5°; the difference between the second and first is 40°. How many degrees in each angle of the triangle?

19. Three men travelled from Los Angeles to Indianapolis and return. Two men shared the lower berth, and the third had an upper. The first two paid in all $223.98, and the third paid $119.64. The difference in price between a lower and an upper is $5.10. What was the cost of one round trip ticket?

20. A man put $12,000 at interest in three sums, the first at 5%, the second at 6%, and the third at $6\frac{1}{2}$%, receiving for the whole $695 per year. The sum at 5% was half as much as the other two sums. Find each of the three sums.

CHAPTER XXIV

EXPONENTS

(In Part Review)

174. Proof of fundamental laws of exponents. The proofs for the four laws governing the use of positive integral exponents are as follows:

I. Proof of the Law of Multiplication,

$$x^a \cdot x^b = x^{a+b}.$$

If a and b are positive integers, we have

$$x^a = x \cdot x \cdot x \cdot x \cdots \text{to } a \text{ factors,}$$

and $$x^b = x \cdot x \cdot x \cdots \text{to } b \text{ factors.}$$

Hence $$\begin{aligned} x^a \cdot x^b &= (x \cdot x \cdot x \cdots \text{to } a \text{ factors}) \times (x \cdot x \cdot x \cdots \\ &\qquad \text{to } b \text{ factors}) \\ &= x \cdot x \cdot x \cdots \text{to } a + b \text{ factors} \\ &= x^{a+b}, \text{ by the definition of an exponent.} \end{aligned}$$

Law I may be stated more completely thus:

$$x^a \cdot x^b \cdot x^c \cdots = x^{a+b+c+\cdots}.$$

II. Proof of the Law of Division,

$$x^a \div x^b = x^{a-b}.$$

Again, if a and b are positive integers, we have

$$x^a \div x^b = \frac{x^a}{x^b} = \frac{x \cdot x \cdot x \cdots \text{to } a \text{ factors}}{x \cdot x \cdot x \cdots \text{to } b \text{ factors}}.$$

If b is less than a, the b factors of the denominator may be canceled with b factors of the numerator, leaving $a - b$ factors in the numerator.

Hence $$\frac{x^a}{x^b} = x^{a-b}.$$

If b is greater than a, a factors of the numerator may be canceled with a factors of the denominator, leaving $b - a$ factors in the denominator.

Hence $$\frac{x^a}{x^b} = \frac{1}{x^{b-a}}.$$

III. **Proof of the Law of Involution**, or raising to a power,

$$(x^a)^b = x^{ab}.$$

As before, if a and b are positive integers, we have

$$\begin{aligned}(x^a)^b &= x^a \cdot x^a \cdot x^a \cdots \text{to } b \text{ factors}\\ &= x^{a+a+a+\cdots}\text{to } b \text{ terms of the exponent}\\ &= x^{ab}.\end{aligned}$$

Law III includes the more general forms:

(*a*) $$(x^a y^b)^c = x^{ac} y^{bc}.$$

(*b*) $$[(x^a)^b]^{c\cdots} = x^{abc\cdots}.$$

When Laws I, II, and III were used in previous work in multiplication and division, we always assumed that a and b were positive integers and, in Law II, that a was greater than b. In the work on radicals (pp. 319–320) the meaning of an exponent was extended so as to include fractional exponents.

IV. **Proof of the Law of Evolution**, or extraction of roots,

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}.$$

Assuming that Law I holds for fractional as well as integral exponents, we have

$$x^{\frac{a}{b}} \cdot x^{\frac{a}{b}} \cdot x^{\frac{a}{b}} \cdots \text{to } b \text{ factors} = x^{b\frac{a}{b}} = x^a.$$

Here the left member consists of b equal factors whose product is x^a. That is, each one of the factors must equal $\sqrt[b]{x^a}$.

This fact will be assumed without proof. We shall now explain the meaning which, according to these laws, must be given to a zero or to a negative exponent.

175. Meaning of zero as an exponent. From Law II we know that

$$y^5 \div y^5 = y^{5-5} = y^0.$$

But $$y^5 \div y^5 = \frac{y^5}{y^5} = 1.$$

Therefore $$y^0 = 1.$$

In general $$y^a \div y^a = y^{a-a} = y^0 = 1.$$

In other words, any expression (not equal to zero), raised to the zero power, becomes equal to 1. Thus, $(5)^0 = (\frac{1}{2})^0 = (-2)^0$ since each is equal to 1. Also, if x is not zero $(3x)^0 = 1$, and if y is not 2, the expression $(y^2 - 4y + 4)^0$ is equal to 1.

176. Meaning of a negative exponent. From Law II

$$x^2 \div x^4 = x^{2-4} = x^{-2}.$$

But $$x^2 \div x^4 = \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Therefore $$\frac{1}{x^2} = x^{-2}.$$

Furthermore $$3^{-3} = \frac{1}{3^3} = \frac{1}{27}, \text{ and}$$

$$m^{-\frac{3}{4}} = \frac{1}{m^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{m^3}}.$$

The above facts may be expressed more generally as follows:

$$y^{-a} = \frac{1}{y^a}.$$

Therefore $$\frac{1}{y^{-a}} = \frac{1}{\frac{1}{y^a}} = y^a.$$

We also have

$$ay^{-b} = \frac{a}{y^b}, \text{ and } \frac{a}{y^{-b}} = ay^b$$

We may therefore say: *In a fraction, any factor of the numerator may be omitted from the numerator and written as a factor of the denominator, and vice versa, provided that the sign of its exponent is changed.*

It follows that any expression involving negative exponents may be written as an expression involving only positive exponents. That is to say, negative exponents are not a mathematical necessity but merely a convenience. The extension of the laws of exponents which brings with it the zero exponent and the negative exponent is an illustration of what is called the Law of Permanence of Form.

It is to be understood that the part of the rule for multiplication (p. 91) and the rule for division (p. 110), which determines the exponents in the product or in the quotient, applies to all numbers, whether positive or negative, integral, fractional, or literal. Hence those rules need not be restated here.

ORAL EXERCISES

Perform the indicated operations and simplify results:

1. $x^{\frac{1}{3}} \cdot x^{\frac{2}{3}}$.
2. $x^{\frac{1}{5}} \cdot x^{\frac{3}{5}}$.
3. $x^{\frac{1}{3}} \cdot x^{\frac{1}{4}}$.
4. $y^{\frac{2}{3}} \cdot y^{\frac{1}{2}}$.
5. $y^{\frac{2}{3}} \div y^{\frac{1}{3}}$.
6. $x^{\frac{2}{5}} \cdot x^{\frac{3}{5}}$.
7. $r^{2} \cdot r^{\frac{1}{2}}$.
8. $r^{2} \div r^{\frac{1}{2}}$.
9. $x^{\frac{1}{3}} \cdot x$.
10. $m^{\frac{2}{3}} \cdot m^{0}$.
11. $m^{\frac{2}{3}} \div m^{0}$.
12. $y^{b} \cdot y^{0}$.
13. $p^{2} \cdot p^{-1}$.
14. $x^{3} \div x^{-1}$.
15. $x^{\frac{1}{3}} \cdot x^{-\frac{2}{3}}$.
16. $x^{\frac{1}{3}} \div x^{-\frac{2}{3}}$.
17. $y^{a} \div y^{-a}$.
18. $x^{0} \cdot x^{\frac{1}{2}}$.
19. $x^{0} \div x^{\frac{1}{2}}$.
20. $2\,m^{0} \cdot (2\,m)^{0}$.
21. $12\,x^{2} \cdot 2\,x^{0}$.
22. $4\,y^{\frac{1}{2}} \div 2\,y^{0}$.
23. $3\,x^{2} \cdot x^{0}$.
24. $2\,y^{\frac{1}{2}} \cdot y^{b}$.
25. $r^{\frac{2}{3}} \div r^{\frac{5}{3}}$.
26. $r^{\frac{1}{2}} \cdot r \cdot r^{\frac{3}{2}}$.
27. $m^{\frac{1}{3}} \cdot m^{\frac{2}{3}} \cdot m^{\frac{1}{3}}$.
28. $y^{0} \cdot y^{4} \cdot y^{-\frac{1}{3}}$.
29. $(x^{\frac{1}{2}} + x^{\frac{1}{3}}) \cdot x^{\frac{1}{4}}$.
30. $m^{2a+1} \cdot m^{3a-2}$.
31. $x^{2+a} \cdot x^{a-2}$.
32. $a^{2} \div a^{-2}$.
33. $y \cdot \sqrt{y}$.
34. $x^{2} \div \sqrt[3]{x}$.
35. $\sqrt{m} \cdot \sqrt[4]{m}$.
36. $\sqrt[3]{y^{5}} \div \sqrt[3]{y^{2}}$.
37. $x \cdot \sqrt{x^{3}}$.
38. $\sqrt{m^{0}} \cdot \sqrt[3]{m^{5}}$.
39. $t^{b-2} \cdot t^{2-b}$.
40. $(x^{3})^{2} \cdot x^{5}$.
41. $(y^{\frac{1}{3}})^{2} \cdot y^{\frac{1}{2}}$.
42. $x^{4b} \div \dfrac{1}{x^{3b}}$.
43. $m^{2} \div \dfrac{1}{m^{-2}}$.
44. $y^{-2} \div \dfrac{1}{y^{-5}}$.
45. $(r^{\frac{1}{2}})^{3} \cdot (r^{3})^{-\frac{1}{2}}$.
46. $by^{2} \div b^{-2}y^{-3}$.

Read the following with positive exponents and simplify the results:

47. r^{-2}.
48. $3\,m^{-4}$.
49. $2 \times y^{-3}$.
50. $8\,p^{2}r^{-4}$.
51. $3\,x^{2}y^{-3}z^{-2}$.
52. $\dfrac{z}{m^{-3}}$.
53. $\dfrac{2\,a}{3\,b^{-2}}$.
54. $\dfrac{3\,x^{0}}{a^{-1}b^{-3}}$.
55. $\dfrac{2\,m^{-2}p^{0}}{r^{-1}}$.
56. $\dfrac{2^{-2}(xy)^{0}}{5^{-3}x^{2}}$.
57. $\dfrac{2\,a^{2}bc^{-3}}{4\,a^{0}b^{-1}}$.

58. $\dfrac{15\,m^2y^{-1}}{5\,ym^{-2}}$.

59. $\dfrac{8^{-1}a^{-2}}{b^{-3}c^{-4}}$.

60. $\dfrac{3\,x^2y^{-5}z^3}{2^{-1}x^{-2}y^2z^0}$.

61. $\dfrac{2^{-3}m^{-2}p^{-4}}{2^{-4}m^{-3}p^{-5}}$.

62. $\dfrac{5\,x^{-2}}{y^{-5}z^2}$.

63. $3\,e^y \div \dfrac{4}{e^{-y}}$.

64. $\dfrac{a^{-2}}{a^{n+1}}$.

65. $(64\,x^{-6})^{-\frac{2}{3}}$.

66. $\dfrac{1}{(9\,a^{-5})^{-\frac{1}{10}}}$.

Simplify and arrange terms so that the exponents occur in the descending order:

67. $a^3 + a^{-2} + 5\,a^{\frac{7}{2}} - 4\,a^0$.

68. $2\,x^{-\frac{1}{4}} + 3\,x^{-3} - x^{\frac{1}{2}} + 2\,x^3$.

69. $m^{\frac{1}{3}} + 2\,m^{-2} - 1 + m^{-\frac{3}{2}} - m^2$.

70. $x + 2\,x^{-\frac{1}{2}} - 3\,x^{\frac{4}{3}} + 2\,x^0 - 5\,x^{\frac{1}{2}}$.

71. $y^2 + \dfrac{1}{y^2} + y^{-\frac{1}{2}} - y^{\frac{1}{2}} + \dfrac{1}{y^0}$.

72. $m^{\frac{2}{3}} + \dfrac{1}{m^2} - 3 - \dfrac{1}{m^{-3}} + \dfrac{1}{m}$.

EXERCISES

Perform the indicated multiplications and write results as integral expressions:

1. $(x + x^{\frac{2}{3}} - x^{\frac{1}{2}})3\,x^{\frac{1}{3}}$.
2. $(x^3 - y^3)x^{-6}y^{-6}$.
3. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})$.
4. $(m^{\frac{1}{2}} - p^{\frac{1}{2}})(m^{\frac{1}{2}} + p^{\frac{1}{2}})$.
5. $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{1}{3}} + y^{\frac{1}{3}})$.
6. $(b^{-4} + 2)(b^{-4} + 3)$.
7. $(y^{3b} - 4\,y^{2b})(y^3 - 3\,y)$.
8. $(m^{-2} + m)^2$.
9. $(x^4 - 3\,x^{-1})^3$.
10. $(4\,z + 2\,z^{-1} - 3\,z^{-2})^2$.
11. $(m^x + m^{-x})^2$.
12. $(m^x - m^{-x})^2$.

13. $(a^9 - 1)^3$.

14. $(x^{-1} + y^{-2})\left(\frac{1}{x} - \frac{1}{y^2}\right)$.

15. $(a^{2y} - 4 + a^{-2y})^2$.

16. $(4\,m^{-\frac{1}{2}} - 3\,m^{\frac{1}{4}})^2$.

17. $(2\,x - 5\,x^{-1} + x^{-2})^2$.

18. $(e^{3x} - 2\,e^{-x})^4$.

19. $(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$.

20. $(p^{\frac{1}{2}} + q^{\frac{1}{2}} + p^{\frac{1}{4}}q^{\frac{1}{4}})(q^{\frac{1}{2}} - p^{\frac{1}{4}}q^{\frac{1}{4}} + p^{\frac{1}{2}})$.

21. $(a + a^{\frac{1}{2}}b^{-\frac{1}{2}} + b^{-1})(a - a^{\frac{1}{2}}b^{-\frac{1}{2}} - b^{-1})$.

EXERCISES

Perform the indicated division and write results as integral expressions:

1. $a^2 \div a^7$.

2. $x^2 \div x^{\frac{1}{2}}$.

3. $y^{\frac{1}{3}} \div y^3$.

4. $ab^{\frac{3}{4}} \div a^{\frac{1}{2}}b^{\frac{1}{4}}$.

5. $\dfrac{xy + x^3y^2}{x^2y^{\frac{1}{3}}}$.

6. $(m^b - 3\,m^{5\,b+2} + 2\,m^{b-1}) \div m^{2\,b-1}$.

7. $(8\,x^{y-4} + 2\,x^{5+y} - 6\,x^{3-y}) \div 2\,x^{y-3}$.

8. $(a - b) \div (a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

9. $(x - y) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

10. $(x + y) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}})$.

11. $(9\,a - 16\,b) \div (3\,a^{\frac{1}{2}} + 4\,b^{\frac{1}{2}})$.

12. $(a^4 + a^2b^{-1} + b^{-2}) \div (a^2 - ab^{-\frac{1}{2}} + b^{-1})$.

13. $(e^y + e^{-y} + 2) \div (e^y + e^{-y})$.

14. $(e^{-3x} + e^{3x} - e^{-x} - e^x) \div (e^{-x} - e^x)$.

15. $(a + 9\,a^{\frac{2}{3}}b^{\frac{2}{3}} + 27\,a^{\frac{1}{3}}b^{\frac{4}{3}} + 27\,b^2) \div (a^{\frac{1}{3}} + 3\,b^{\frac{2}{3}})$.

16. $(a^4 + 6\,a^{-2} + 6\,a^2 + a^{-4} + 11) \div (a^{-2} + a^2 + 3)$.

17. $(9\,x^{5n-4} - x^{3n-2} + 2\,x^{2n-1}) \div (2\,x^{n-1} + 3\,x^{2n-2})$.

Divide:

18. $a^{\frac{4}{3}} - b^{\frac{4}{3}} + a^{\frac{1}{3}}b - ab^{\frac{1}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

19. $3\,r^{-5} + r^3 - 4\,r^{-3}$ by $2\,r^{-1} + r + 3\,r^{-3}$.

20. $x^{\frac{3}{5}} - m^{\frac{3}{5}}$ by $[(x^{\frac{1}{5}} - m^{\frac{1}{5}}) \div (x^{\frac{1}{10}} + m^{\frac{1}{10}})]$.

21. $40 - 12\,a - 10\,a^{-1} - 4\,a^{-\frac{1}{2}} - 14\,a^{\frac{1}{2}}$ by $2\,a^{\frac{1}{2}} + 3 - 5a^{-\frac{1}{2}}$.

22. $93 - 10(x^{-2a} + x^{2a}) - 24(x^{-a} + x^{a})$ by $2\,x^{a} - 5\,x^{-a} + 8$.

NOTE. To us, who use the notation of exponents every day, it seems so simple and natural a method of expressing the product of several factors that it is difficult to understand why such a long time was necessary to develop it. But here, as in many other instances, it required a great man to discover what to us seems the most obvious relation. The man who brought the notation of exponents to its modern form was John Wallis (1616–1703), an Englishman.

Though the idea of using negative and fractional exponents had occurred to writers before Wallis, it was he who showed their naturalness, and who introduced them permanently. He also was the first to use the ordinary sign ∞ to denote infinity.

MISCELLANEOUS EXERCISES

Find the numeric value of the following:

1. 2^{-1}.

2. 3^{-3}.

3. 5^{-2}.

4. $2^{-3} \cdot 4^0$.

5. $3^{-4} \cdot 2^{-1}$.

6. $2^{-5} \cdot 5^2 \cdot 0$.

7. $(\frac{1}{3})^{-1}$.

8. $(\frac{1}{2})^{-2}$.

9. $(\frac{2}{3})^{-3} \cdot (2)^0$.

10. $\dfrac{2^{-3} \cdot 5^{-2}}{10^{-2}}$.

11. $\dfrac{1}{243^{-\frac{3}{5}}}$.

12. $0^2 \cdot 3^{-0}$.

13. $9^{-\frac{3}{2}}$.

14. $27^{-\frac{2}{3}}$.

15. $81^{-\frac{1}{4}}$.

16. $16^{-\frac{3}{4}}$.

17. $25^{-\frac{3}{2}}$.

18. $16^{.5}$.

19. $81^{1.25}$.

20. $(-8)^{-\frac{1}{3}}$.

21. $(9)^{-\frac{3}{2}}$.

22. $(2)^{-4}$.

23. $-\ 64^{-\frac{5}{6}}$.

24. $\sqrt[4]{16^{-3}}$.

25. $-\sqrt[3]{-\ 27^{2}}$.

26. $(\frac{1}{2})^{-3}\cdot(\frac{1}{3})^{0}\cdot(\frac{1}{4})^{\frac{3}{2}}$.

27. $(.25)^{\frac{1}{2}}$.

28. $(.09)^{-\frac{3}{2}}$.

29. $(.0016)^{\frac{3}{4}}$.

30. $\dfrac{\sqrt[4]{2^{-4}}\cdot\sqrt{2^{-2}}}{2^{-2}}$.

Write with positive exponents and simplify the results:

31. $\dfrac{3}{m^{-2}+b^{-2}}$.

32. $\dfrac{3\,a^{2}b}{a^{-3}-b^{-2}}$.

33. $\dfrac{a^{-2}}{b^{-2}+c^{-2}}$.

34. $\dfrac{x^{-3}-8\,m^{-3}}{x^{-1}-2\,m^{-1}}$.

35. $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$.

36. $\dfrac{m^{-3}+27\,a^{-3}}{m^{-1}+3\,a^{-1}}$.

Write without a denominator and simplify the results:

37. $\dfrac{3\,mb}{x^{2}}$.

38. $\dfrac{11\,x^{2}}{y}$.

39. $\dfrac{2\,r}{3\,s^{-5}b^{2}}$.

40. $\dfrac{3\,a^{-1}b}{3^{-1}ab^{3}}$.

41. $\dfrac{15\,m^{-2}n^{-1}}{5\,m^{2}n}$.

42. $\dfrac{2\,ab^{-2}}{(a-b)^{2}}$.

43. $\dfrac{6\,m^{3}(n-1)}{(n-1)^{-\frac{1}{3}}}$.

44. $\dfrac{2}{6\,a^{2}(a+b^{2})^{-\frac{1}{2}}}$.

45. $\dfrac{2(x+y)^{2}}{3\,m(x-y)^{-\frac{3}{2}}}$.

46. $\dfrac{5(a+b^{2})^{-3}}{2(a^{-2}+b)^{2}}$.

47. $\dfrac{(18\,x^{-y})\cdot(y^{-2x})}{36\,x^{-2}y^{-3x}}$.

48. $\dfrac{p^{-2}q^{-1}}{p^{2}q^{-4}(p^{2}-q^{-2})^{\frac{1}{2}}}$.

Simplify:

49. $(16\,a^{12}b^{8})^{-\frac{1}{4}}$.

50. $(3\,a^{10})^{0}\cdot 8\cdot 16^{-\frac{1}{2}}$.

51. $(p^{2}q)^{-3}(p-q)$.

52. $(m^{2})^{3b}\cdot m^{3b}$.

53. $(m^{2x})^{3y} \cdot (m^{2y})^{3x}$.

54. $(x^{a+2})^{2} \cdot (x^{2-a})^{2}$.

55. $(y^{2})^{a+b}(y^{3})^{a-b}$.

56. $(p^{2}q)^{m}(pq^{2})^{3m}$.

57. $(a^{-2} + a^{2})a^{2}$.

58. $(x^{2} - 3\,xy^{-1})x^{-2}$.

59. $2^{3} \cdot 8^{2} = 2^{?}$.

60. $2^{a} \cdot 4^{b} = 2^{?}$.

61. $\dfrac{3^{x} \cdot 9^{2x}}{27^{4x}} = 3^{?}$.

62. $2^{m} \cdot 4^{3m} \div 2^{2n} = 2^{?}$.

63. $\dfrac{3^{a+2}}{9^{a}(3^{3a-1})^{a}} \div \dfrac{9^{a+1}}{(27^{2a-3})^{a+1}} + 3 = ?$

Solve for x:

64. $x^{\frac{1}{2}} = 3$.

65. $x^{\frac{3}{2}} = 4$.

66. $x^{-\frac{1}{2}} = 3$.

67. $x^{-\frac{1}{4}} = 2$.

68. $x^{\frac{2}{3}} = 9$.

69. $x^{-\frac{3}{4}} = 8$.

70. $x^{\frac{1}{4}} = -2$.

71. $x^{\frac{4}{3}} = 81$.

72. $x^{-\frac{2}{3}} = \frac{1}{4}$.

73. $x^{-\frac{3}{2}} = 216$.

74. $\frac{1}{4}\,x^{-\frac{2}{3}} = 1$.

75. $(x^{-\frac{1}{2}})^{-2} = 2$.

76. $(a^{\frac{1}{2}}x^{\frac{1}{3}})^{-6} = 9$.

77. $\dfrac{\sqrt[4]{x^{\frac{2}{3}}}}{\sqrt[4]{x^{\frac{1}{3}}}} = \dfrac{\sqrt[3]{4}}{\sqrt[3]{9}}$.

CHAPTER XXV

SQUARE ROOT

(In Part Review)

177. Square root. The **square root** of any number is one of the two equal factors whose product is the number.

From the law of signs in multiplication it follows that

Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.

Thus $$\sqrt{a} \cdot \sqrt{a} = (-\sqrt{a})(-\sqrt{a}) = a.$$

178. Square root of a monomial. For extracting the square roots of any monomial we have the

Rule. *Write the square root of the numeric coefficient preceded by the double sign ± and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule similar to the preceding one holds for the fourth root, the sixth root, and other even roots.

179. Cube root. The cube root of any number is one of the three equal factors whose product is the number.

For extracting the cube root of a monomial we have the

Rule. *Write the cube root of the numeric coefficient preceded by the sign of the monomial followed by all the letters of the monomial, giving to each letter an exponent equal to one third of its exponent in the monomial.*

A rule similar to the preceding one holds for the fifth root, the seventh root, and other odd roots.

180. Principal root. For a given index (see section 120) the **principal root** of a number is its one real root if it has but one, or its positive real root if it has two real roots.

Then the principal square root of 9 is $+3$; the principal fourth root of 16 is $+2$, not -2. The square root of -4 or -9 is *not* a real number; a negative number has no principal square root.

The principal cube root of 8 is 2, of -27 is -3. The principal fifth root of 32 is $+2$, of -32 is -2.

Every number has more than one root of given odd index. The number 8, for example, has two other cube roots besides its principal cube root 2. What they are and how they are obtained will be made clear in the chapter on Imaginaries, where the consideration of the square roots of negative numbers is taken up.

Only the principal odd root of a number will be considered in the following exercises.

ORAL EXERCISES

Find the principal square root of the following:

1. 9.
2. 36.
3. x^2.
4. $25\,b^4$.
5. $49\,m^6$.
6. $36\,r^4$.
7. $4\,p^{12}$.
8. $16\,x^8$.
9. $\frac{1}{9}$.
10. $\frac{1}{x^2}$.
11. $\frac{4}{m^4}$.
12. $\frac{a^2}{b^2}$.
13. a^{-2}.
14. $4\,x^{-4}$.
15. $25\,a^2b^{-2}$.
16. $100\,p^0$.
17. $x^{\frac{2}{3}}$.
18. $m^{\frac{4}{5}}$.

Find the principal cube root of the following:

19. 8.
20. 27.
21. 125.
22. $8x^3$.
23. $64y^6$.
24. $8a^{12}$.
25. -8.
26. -64.
27. -27.
28. $-125x^6$.
29. $-8b^3$.
30. $-64z^6$.
31. $216x^3$.
32. $343x^6$.
33. $512m^3$.
34. $-64x^{\frac{3}{2}}$.
35. $-8a^{-3}$.
36. $-125m^{-12}$.

Find the principal fourth root of the following:

37. 81.
38. 16.
39. 625.
40. r^4.
41. x^8.
42. m^{12}.
43. $16a^4$.
44. $625b^{-4}$.
45. $81c^{-12}$.
46. $m^{-\frac{4}{3}}$.
47. $r^{-\frac{8}{5}}$.
48. $y^{-\frac{4}{7}}$.

Give the principal root and one other root for the following:

49. The fourth root of 16; of x^8; of x^{-12}.

50. The sixth root of 64; of a^{12}; of a^{-12}.

51. What is the sign of the principal *odd* root of a positive number? the principal odd root of a negative number?

52. What is the sign of the principal even root of a positive number?

53. State a rule for extracting the fourth root of a monomial.

54. State a rule for extracting the fifth root of a monomial.

55. Can one obtain the fifth root of a monomial by extracting the square root of its cube root? by extracting the cube root of its square root? Explain.

181. Square root of polynomials. Review the examples and the rule on pages 288–289 for extracting the square root of an algebraic expression.

EXERCISES

Extract the square roots of the following:

1. $9x^4 - 12x^3 + 16x^2 - 8x + 4$.
2. $m^6 + 6m^4 + 4m^3 + 9m^2 + 4 + 12m$.
3. $16p^6 + 24p^5 - 80p^3 + 9p^4 + 100 - 60p^2$.
4. $4t^4 - 20a^2t^2 + 12t^2 + 25a^4 + 9 - 30a^2$.
5. $16a^{-4} - 24a^{-2} - 30a^2 + 25a^4 + 49$.
6. $625 + 9a^2 - 6a^{\frac{3}{2}} - 149a + 50a^{\frac{1}{2}}$.
7. $81x^3 + 36x^{\frac{3}{2}} - 54x + 4 - 12x^{-\frac{1}{2}} + 9x^{-1}$.
8. $4x^3 - 8x^{\frac{3}{2}} + 20x + 4 - 20x^{-\frac{1}{2}} + 25x^{-1}$.
9. $144r^{16} + 9r^{-10} + 100r^{-4} - 72r^3 - 240r^6 + 60r^{-7}$.
10. $\dfrac{9x^2}{4y^2} + \dfrac{4y^2}{9x^2} - \dfrac{15x}{y} + 27 - \dfrac{20y}{3x}$.

HINT. Reduce the expression to a single fraction and find the square root of the numerator and denominator separately.

11. $4x^4 - 12x^3 + \frac{1}{4} - 3x + 11x^2$.
12. $4x^6 - 4x - \frac{1}{2}x^{-2} + x^3 + \frac{1}{16} + x^{-4}$.
13. $\dfrac{9a^4}{25} - \dfrac{4a^3}{5} - \dfrac{10a}{3} + \dfrac{25}{4} + \dfrac{31a^2}{9}$.
14. $\frac{4}{9}a^6 - \frac{2}{3}a^4 + \frac{8}{3}a^3 + \frac{1}{4}a^2 + 4 - 2a$.
15. $4m^2 + \dfrac{1}{m^2} - \dfrac{10}{m} + 21 + 20m$.
16. $\dfrac{x^2}{4} + \dfrac{4}{x^2} + 3 - x - \dfrac{4}{x}$.

17. $2 + \frac{a^2}{b^2} + \frac{4a}{b} + \frac{b^2}{a^2} - \frac{4b}{a}$.

18. $\frac{c^2}{d^2} + c^2d^2 + \frac{d^2}{c^2} + 2 - 2c^2 - 2d^2$.

19. $\frac{1}{4y^2} - \frac{3}{y} + \frac{2}{3x} + 9 - \frac{4y}{x} + \frac{4y^2}{9x^2}$.

20. $\frac{4a^2}{9b^2} + 1\frac{8}{9} + \frac{4a}{3b} + \frac{4b^2}{9a^2} + \frac{4b}{3a}$.

Find the first four terms in the square root of the following:

21. $3 + 5x$.

22. $x + 2$.

23. $x^3 + x$.

24. $\frac{16}{25} + m^2$.

182. Square root of arithmetic numbers. Review pages 291–295 for explanation, examples and rule for extracting the square root of arithmetic numbers.

EXERCISES

Find the positive square roots of the following:

1. 2,401.

2. 5,184.

3. 18,496.

4. 59,049.

5. 67.24.

6. 1.5876.

7. 231.04.

8. 585,225.

9. 7,387,524.

10. 3,519,376.

11. 7.7284.

12. .00018496.

Find correct to three decimal places the positive square roots of the following:

13. 6.

14. .85.

15. .08754.

16. .998765.

17. $\frac{6}{11}$.

18. $3\frac{2}{3}$.

19. $\frac{12}{7}$.

20. $\frac{5}{7}$.

21. $14\frac{3}{8}$.

22. $78\frac{1}{6}$.

23. Find the hypotenuse of a right triangle whose sides are 15 feet and 112 feet.

24. A baseball diamond is 85 feet on a side. What is the distance from first base to third base, to the nearest hundredth of a foot?

25. A right triangle has a hypotenuse 75 feet and one side 54 feet. Find the remaining side to the nearest hundredth of a foot.

26. The hypotenuse and one side of a right triangle are respectively 3569 feet and 2576 feet. Find the remaining side to the nearest tenth of a foot.

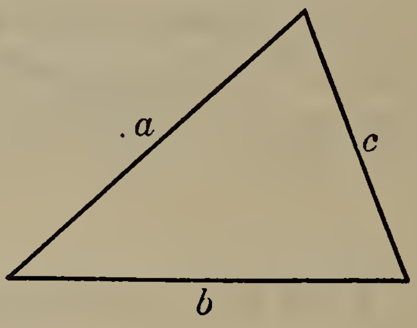
27. The side of an equilateral triangle is 21 inches. Find the altitude to the nearest tenth of an inch.

28. Find the side of an equilateral triangle whose altitude is 24 inches, to the nearest hundredth of an inch.

Fact from Geometry. Area of triangle

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c = the sides of the triangle and $s = \frac{1}{2}$ the perimeter of the triangle.



29. A triangle has sides 12 inches, 14 inches, and 16 inches. What is its area to the nearest hundredth of a square inch?

30. By the method of Exercise 29 find to the nearest hundredth of a square inch the area of a triangle, each side of which is 16 inches.

31. Find correct to two decimal places the sum of all the diagonal lines that can be drawn on the faces of a cube whose side is 21 inches.

32. Find to the nearest thousandth of an inch the radius of a circle whose area is 432 square inches.

33. A room is 18 feet by 21 feet by 12 feet. What is the diagonal of the room to the nearest tenth of a foot?

34. Find to the nearest thousandth of an inch the diagonal of a cube 2 inches on a side.

35. A cube has an outside area of 17,592 square inches. Find the volume to the nearest tenth of a cubic inch.

Using the table for squares and square roots at the end of the book:

36. Find the square roots of 8; 28; 45; 84; 57. Compare the results found by computing the required square roots to three decimal places.

37. Find the diagonals of the squares whose sides are 3; 7; 6.

38. Find the hypotenuse of the right triangles whose sides are 5 feet and 7 feet; 3 inches and 8 inches; 2 rods and 9 rods.

CHAPTER XXVI

RADICALS

(In Part Review)

183. Definitions. Review the discussion on pages 302, 303, and 319.

184. Irrational numbers. Any real number which is not rational is **irrational**. (See section 186.)

If a number under a radical sign is such that the root indicated cannot be exactly obtained, the radical represents an **irrational** number.

For example, $\sqrt{5}$ and $\sqrt[3]{7}$ are irrational. Approximate values for these are given in the table at the end of the book.

A repeating decimal, though endless, is not an irrational number, for any repeating decimal can be expressed as a common fraction, and is therefore rational.

Thus the repeating decimal .272727⋯ is not irrational, as it exactly equals $\frac{3}{11}$. Similarly, .2857142857142⋯ exactly equals $\frac{2}{7}$, etc.

NOTE. The number $\frac{2}{7}$ reduced to a decimal repeats the digits in groups of six each, and the mere fact that a decimal does so repeat is proof that it is a rational number. On the other hand, the number π is known to be irrational, and its value has been computed to 707 decimals, showing no repetition. The fact that it does not repeat in 700 digits is, however, no proof that π is irrational, for decimals with even more than that many digits do repeat. For example, the fraction $\frac{10000}{7699}$ equals the decimal 1.29^{+}, which repeats in groups of 7698 digits each.

185. Imaginary. An indicated *square* root of a *negative* number is called an **imaginary** number.

Thus $\sqrt{-9}$, $\sqrt{-5}$, and $\sqrt{-15}$ are imaginary numbers. And $5+\sqrt{-1}$ is also imaginary, though, as will be seen later (Chapter XXIX), such numbers are usually called complex numbers.

186. Classification of numbers. All the numbers of algebra then may be placed in one or the other of two classes: **real** numbers and **imaginary** numbers.

Real numbers, as we have seen, are of two kinds, **rational** numbers and **irrational** numbers.

187. Surd. A **surd** is an irrational number in which the radicand is rational.

Thus $\sqrt{2}$, $\sqrt[3]{5}$, etc., are surds. But $\sqrt{5+\sqrt{2}}$ and $\sqrt{\pi}$ are not surds.

188. The algebraic sign of a radical. The square root of 9 is both $+3$ and -3. The symbol $\sqrt{9}$, however, signifies only $+3$, the principal root (section 122). Similarly, $\sqrt[4]{81}$ is $+3$ and $\sqrt[6]{64}$ is $+2$. But $-\sqrt{9}$ is -3, and $-\sqrt[4]{16}$ is -2. The symbol $\pm\sqrt{36}$ denotes both $+6$ and -6. Further, $+\sqrt[3]{27}=+3$, $-\sqrt[3]{27}=-3$, and $-\sqrt[3]{-27}=+3$.

The foregoing remarks apply also to fractional exponents. Thus $4^{\frac{1}{2}}=+2$ only, and $81^{\frac{1}{4}}=+3$ only. It should be noted that these statements really define the meaning of such symbols as $\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$. Such an understanding as this avoids all the ambiguity which would arise if $\sqrt{16}$ meant both $+4$ and -4. The distinctions here made are especially needed in checking irrational equations (Chapter XXVIII).

ORAL EXERCISES

Find the numeric value of the following:

1. $\sqrt{9}$.
2. $-\sqrt{4}$.
3. $\sqrt{25}$.
4. $\sqrt[3]{8}$.
5. $\sqrt[3]{-27}$.
6. $-\sqrt[4]{625}$.
7. $-\sqrt[4]{16}$.
8. $\sqrt[5]{243}$.
9. $-\sqrt[6]{64}$.
10. $-\sqrt[3]{-125}$.
11. $4^{\frac{1}{2}}$.
12. $27^{\frac{1}{3}}$.
13. $9^{\frac{3}{2}}$.
14. $81^{\frac{1}{4}}$.
15. $8^{\frac{2}{3}}$.
16. $(\frac{1}{4})^{\frac{1}{2}}$.
17. $(\frac{1}{25})^{\frac{1}{2}}$.
18. $(-125)^{\frac{1}{3}}$.
19. $(36)^{\frac{3}{2}}$.
20. $(125)^{\frac{4}{3}}$.
21. $16^{\frac{5}{4}}$.

Read as if expressed in radical form:

22. $x^{\frac{2}{3}}$.
23. $y^{\frac{3}{5}}$.
24. $(xy)^{\frac{3}{2}}$.
25. $(5m)^{\frac{1}{2}}$.
26. $5m^{\frac{1}{3}}$.
27. $2mx^{\frac{1}{4}}$.
28. $3m^{\frac{1}{2}}p^{\frac{1}{3}}$.
29. $6ab^{\frac{2}{3}}$.
30. $x(x-1)^{\frac{1}{3}}$.
31. $3b(2a-5)^{\frac{2}{5}}$.
32. $y^{\frac{1}{a}}$.
33. $3^{\frac{1}{2}}z^{\frac{1}{b}}$.
34. $x^{\frac{a}{b}}y^{\frac{3b}{a}}$.
35. $x^{\frac{m}{2}}y^{\frac{3m}{4}}$.

Read as if expressed with fractional exponents:

36. $\sqrt{x^3}$.
37. $\sqrt{ab^5}$.
38. $\sqrt[3]{x^4}$.
39. $\sqrt[3]{m^2x^7}$.
40. $3\sqrt{m}$.
41. $\sqrt[3]{2z^2}$.
42. $\sqrt[3]{(m+n)^4}$.
43. $\sqrt{(2x+y)^3}$.
44. $\sqrt[3]{m^4(x-2)}$.

45. What are the two square roots of 25?

46. What are two fourth roots of 81? of 256? of 625?

47. What are the numeric values of $\sqrt[4]{81}$? of $\sqrt[4]{256}$? of $\sqrt[4]{625}$?

48. What are two sixth roots of 729?

49. What is the distinction between a rational number and an irrational one?

50. Which of the numbers 4, $\frac{1}{2}$, 365, $\sqrt{9}$, $\sqrt{5}$, and π are rational? Which are irrational?

51. Give a geometric illustration of an irrational number by means of a right triangle.

52. Is a radical always a surd? Illustrate.

53. Is a surd always a radical? Illustrate.

54. Distinguish between a surd and a radical.

55. Which of the numbers $\sqrt{2}$, $\sqrt{4}$, $\sqrt[3]{8}$, $\sqrt{\sqrt[3]{5}}$, $\sqrt{2+\sqrt{2}}$ and $\sqrt{2\pi}$ are surds? Which are radicals?

56. What is the principal square root of 9; the principal cube root of 27; the principal cube root of -8?

57. Name the order of $\sqrt{3}$; of $a^{\frac{1}{4}}$; of $\sqrt[3]{6}$; of $a^{\frac{3}{4}}$; of $\sqrt[3]{a^2}$.

58. Give an example of (*a*) a real number; (*b*) an imaginary number; (*c*) a rational number; (*d*) an irrational number; (*e*) a radical; (*f*) a radicand; (*g*) an index; (*h*) a surd; (*i*) the principal odd root of a positive number; (*k*) the principal even root of a positive number; (*l*) the principal odd root of a negative number.

189. Simplification of radicals. The form of a radical expression may be changed without altering its numeric value. It is often desirable to change the form of a radical so that its numeric value can be computed with the least possible labor.

The simplification of a radical is based on the general identity

$$\sqrt[n]{a^n b} = \sqrt[n]{a^n} \cdot \sqrt[n]{b} = a\sqrt[n]{b}.$$

A radical is in its simplest form when the radicand

I. *Is integral.*

II. *Contains no rational factor raised to a power which is equal to, or greater than, the order of the radical.*

III. *Is not raised to a power, unless the exponent of the power and the index of the root are prime to each other.*

For the meaning of I, II, and III study carefully the following

EXAMPLES

Examples of I:

1. $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9} \cdot 6} = \sqrt{\frac{1}{9}} \cdot \sqrt{6} = \frac{1}{3}\sqrt{6}.$

2. $6\sqrt{\frac{1}{2}} = 6\sqrt{\frac{2}{4}} = 6\sqrt{\frac{1}{4} \cdot 2} = 6\sqrt{\frac{1}{4}} \cdot \sqrt{2} = 6 \cdot \frac{1}{2}\sqrt{2} = 3\sqrt{2}.$

3. $\sqrt{\frac{5}{7\,a}} = \sqrt{\frac{35\,a}{49\,a^2}} = \sqrt{\frac{1}{49\,a^2} \cdot 35\,a} = \frac{1}{7\,a}\sqrt{35\,a}.$

Examples of II:

1. $\sqrt{9\,x^3} = \sqrt{9\,x^2 \cdot x} = \sqrt{(3\,x)^2 \cdot x} = 3\,x\sqrt{x}.$

2. $3\sqrt[3]{54\,m^7} = 3\sqrt[3]{27\,m^6 \cdot 2\,m} = 3\sqrt[3]{(3\,m^2)^3 \cdot 2\,m}$
$= 9\,m^2\sqrt[3]{2\,m}.$

3. $\sqrt{18 - 9\sqrt{3}} = \sqrt{9(2 - \sqrt{3})} = 3\sqrt{2 - \sqrt{3}}.$

Examples of III:

1. $\sqrt[4]{9} = \sqrt[4]{(3)^2} = 3^{\frac{2}{4}} = 3^{\frac{1}{2}} = \sqrt{3}.$

2. $\sqrt[12]{8} = \sqrt[12]{(2)^3} = 2^{\frac{3}{12}} = 2^{\frac{1}{4}} = \sqrt[4]{2}.$

3. $\sqrt[4]{x^2y^6} = \sqrt[4]{(xy^3)^2} = (xy^3)^{\frac{2}{4}} = (xy^3)^{\frac{1}{2}} = y\sqrt{xy}.$

A radical of the second order is simplified by the use of the

Rule. *Separate the radicand into two factors one of which is the greatest perfect square which it contains. Then take the square root of this factor and write it as the coefficient of the radical of which the other factor is the radicand.*

If the original radical has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for simplifying radicals involving the cube root and roots of higher orders.

EXERCISES

(Check the values obtained in Exercises 1, 2, 3, 6, 7, 11, and 12 by using the table at the end of the book.)

Simplify:

1. $\sqrt{8}$.
2. $\sqrt{20}$.
3. $\sqrt{75}$.
4. $\sqrt{128}$.
5. $\sqrt{108}$.
6. $\sqrt{60}$.
7. $\sqrt{99}$.
8. $\sqrt{125}$.
9. $\sqrt{432}$.
10. $\sqrt{1250}$.
11. $3\sqrt{32}$.
12. $\sqrt[3]{24}$.
13. $8\sqrt[3]{162}$.
14. $\sqrt[4]{405}$.
15. $2\sqrt[5]{729}$.
16. $\sqrt{x^3}$.
17. $\sqrt{ab^3}$.
18. $\sqrt[3]{m^2n^4}$.
19. $2\sqrt[3]{54\,a^5}$.
20. $\sqrt{\frac{1}{5}}$.

Solution.
$$\sqrt{\tfrac{1}{5}} = \sqrt{\tfrac{5}{25}} = \sqrt{\tfrac{1}{25}\cdot 5} = \tfrac{1}{5}\sqrt{5}.$$

21. $\sqrt{\frac{1}{2}}$.
22. $\sqrt{\frac{3}{5}}$.
23. $\sqrt[3]{\frac{3}{4}}$.
24. $\sqrt{\dfrac{1}{x}}$.
25. $\sqrt{\dfrac{1}{2\,a}}$.
26. $\sqrt{\dfrac{2}{x^3}}$.
27. $\sqrt[3]{\dfrac{1}{mn^3}}$.
28. $\sqrt{\frac{5}{27}}$.
29. $\sqrt[3]{-\frac{4}{9}}$.
30. $\sqrt{(2)^2-(\frac{3}{5})^2}$.
31. $\sqrt{a^2-\left(\dfrac{a}{3}\right)^2}$.

32. $\sqrt{\left(\frac{x+4}{4}\right)^2 - x}$.

33. $\sqrt{\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1}$.

34. $\sqrt{9 - 18\sqrt{5}}$.

HINT. $\sqrt{9 - 18\sqrt{5}} = \sqrt{9(1 - 2\sqrt{5})}$.

35. $\sqrt{16 + 8\sqrt{3}}$.

36. $\sqrt{m^3 + m^2\sqrt{2}}$.

37. $\sqrt[3]{250 - 125\sqrt{5}}$.

38. $\sqrt{p^2 + 2\,p^2\sqrt{10}}$.

39. $\sqrt{\frac{L^2 + L^2\sqrt{3}}{2}}$.

40. $\sqrt[6]{8}$.

Solution. $\sqrt[6]{8} = \sqrt[6]{(2)^3} = 2^{\frac{3}{6}} = 2^{\frac{1}{2}} = \sqrt{2}$.

41. $\sqrt[4]{25}$.

42. $\sqrt[6]{9}$.

43. $\sqrt[4]{x^2y^6}$.

44. $\sqrt[4]{9\,a^{10}}$.

45. $\sqrt[4]{36\,x^2y^4}$.

46. $\sqrt[6]{4\,b^2}$.

47. $\sqrt[4]{\frac{r^2}{16}}$.

48. $\sqrt[6]{\frac{49\,b^6}{36}}$.

Express entirely under one radical sign:

49. $4\sqrt{3}$.

Solution. $4\sqrt{3} = \sqrt{16} \cdot \sqrt{3} = \sqrt{48}$.

50. $2\sqrt{5}$.

51. $4\sqrt[3]{5}$.

52. $x\sqrt{x}$.

53. $3\,a\sqrt[3]{a^2}$.

54. $9\sqrt[3]{\frac{1}{9}}$.

55. $\frac{x}{4}\sqrt[3]{\frac{16}{x^2}}$.

56. $e^x\sqrt{e^x - e^{-x}}$.

57. $(b+2)\sqrt{\frac{1}{b^2 - 4}}$.

58. $\frac{a - 2\,b}{3}\sqrt[3]{\frac{27}{(a - 2\,b)^2}}$.

59. $\frac{x + 5\,y}{2}\sqrt[5]{\frac{32\,x}{2(x + 5\,y)^4}}$.

Express in simplest form, with one radical sign:

60. $\sqrt{\sqrt{3}}$.

Solution. $\sqrt{\sqrt{3}} = \sqrt{3^{\frac{1}{2}}} = 3^{\frac{1}{4}} = \sqrt[4]{3}$.

61. $\sqrt{\sqrt{a}}$.

62. $\sqrt[3]{\sqrt{x}}$.

63. $\sqrt{\sqrt{x^3}}$.

64. $\sqrt{\sqrt[3]{y}}$.

65. $\sqrt[3]{\sqrt[3]{z}}$.

66. $\sqrt{2\sqrt{2}}$.

67. $\sqrt{5\sqrt{5\sqrt{5}}}$.

68. $\sqrt[4]{\sqrt{27}}$.

69. $2\sqrt[3]{3\sqrt[3]{5}}$.

70. $\sqrt[b]{\sqrt[a]{w^c}}$.

71. $\sqrt[\frac{m}{n}]{x^{\frac{m}{a}}}$.

Find by the formula of Exercise 29, page 416, the areas of the triangles whose sides are:

72. 3, 4, and 5.

73. 13, 14, and 15.

74. 17, 25, and 28.

75. 92, 117, and 205.

190. Addition and subtraction of radicals. *Similar radicals* are radicals of the same order, with radicands which are identical or which can be made so by simplification.

The sum or the difference of similar radicals can be expressed as one term, while the sum or difference of dissimilar radicals can only be indicated.

EXERCISES

Simplify and collect:

1. $\sqrt{3}+\sqrt{27}$.

Solution. $\sqrt{3}+\sqrt{27}=\sqrt{3}+3\sqrt{3}=4\sqrt{3}$.

2. $\sqrt{\frac{1}{2}}+2\sqrt{8}$.

3. $\sqrt{48}+\sqrt{75}-\sqrt{12}$.

4. $2\sqrt{18}-5\sqrt{50}+3\sqrt{98}$.

5. $3\sqrt{20}+12\sqrt{45}+2\sqrt{125}$.

6. $7\sqrt{108}+4\sqrt{3}-10\sqrt{27}$.

7. $3\sqrt[3]{2} - \sqrt[3]{16} + 5\sqrt[3]{54}.$

8. $2\sqrt[3]{375} + \sqrt[3]{24} - \sqrt[3]{648}.$

9. $3\sqrt[3]{54} - \sqrt[3]{2000} - \sqrt[3]{250}.$

10. $8\sqrt{\frac{2}{3}} - 2\sqrt{\frac{3}{2}} + 5\sqrt{\frac{1}{6}}.$

11. $2\,a\sqrt[8]{a^4} - 3\,a\sqrt[6]{a^3} + 9\sqrt{a^3}.$

12. $\sqrt{\dfrac{2\,a}{b}} - \sqrt{\dfrac{2\,b}{a}} + \sqrt{\dfrac{a}{2\,b}}.$

13. $\sqrt{\dfrac{5\,a^3}{x}} + \sqrt{\dfrac{a}{5\,x^3}} - \sqrt{\dfrac{a^5}{5\,x^7}}.$

14. $\sqrt[4]{162\,x^9} - \sqrt[4]{32\,x^5} + \sqrt[4]{512\,x} - \sqrt[4]{2\,x}.$

15. $3\,c\sqrt{(a+b)^3} + c\sqrt[6]{(a+b)^3} + \sqrt[4]{(a+b)^2}.$

16. $\sqrt[3]{(x-y)^4} + m\sqrt[6]{x^2 - 2\,xy + y^2} + (x+y)\sqrt[3]{(x-y)}.$

17. $(x-y)\sqrt{\dfrac{x+y}{x-y}} + \sqrt{9\,x^2 - 9\,y^2} + \dfrac{x+y}{x-y}\sqrt{\dfrac{25\,xy^2 - 25\,y^3}{x+y}}.$

18. $\sqrt{\dfrac{3}{x}} + \sqrt{\dfrac{x}{3}} + \sqrt{\dfrac{x^2+9}{3x} + 2} - \sqrt{\dfrac{x^2+9}{3x} - 2}.$

191. Multiplication of real radicals. Real radicals of the same order are multiplied as follows:

EXAMPLE 1

Multiply $3\sqrt{v} + 2\sqrt{x} - 5\sqrt{vx}$ by $3\sqrt{vx}$.

Solution.

$$\begin{array}{l} 3\sqrt{v} + 2\sqrt{x} - 5\sqrt{vx} \\ \underline{3\sqrt{vx}\qquad\qquad\qquad\quad} \\ 9\,v\sqrt{x} + 6\,x\sqrt{v} - 15\,vx \end{array}$$

Real radicals of different order are multiplied as follows:

EXAMPLE 2

Multiply $\sqrt{m}$ by $\sqrt[3]{v}$.

Solution. $\sqrt{m} = m^{\frac{1}{2}} = m^{\frac{3}{6}} = \sqrt[6]{m^3}.$

$\sqrt[3]{v} = v^{\frac{1}{3}} = v^{\frac{2}{6}} = \sqrt[6]{v^2}.$

Then $\sqrt{m} \cdot \sqrt[3]{v} = \sqrt[6]{m^3} \cdot \sqrt[6]{v^2} = \sqrt[6]{m^3v^2}.$

The method of multiplying real radicals is stated in the

RULE. *If necessary, reduce the radicals to the same order.*

Find the products of the coefficients of the radicals for the coefficient of the radical part of the result.

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for the multiplication of imaginary numbers. This case is discussed in Chapter XXIX.

EXERCISES

Perform the indicated multiplications and simplify the products:

1. $\sqrt{2} \cdot \sqrt{8}.$
2. $\sqrt{20} \cdot \sqrt{10}.$
3. $\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{2}{3}}.$
4. $\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{4}{3}}.$
5. $(\sqrt{3} - 3\sqrt{2})\sqrt{3}.$
6. $(\sqrt{2} - \sqrt{3})(2\sqrt{2} + 5\sqrt{3}).$
7. $(\sqrt{a} - 5\sqrt{ab})(2\sqrt{a} + 3\sqrt{ab}).$
8. $(4\sqrt{2} - 5\sqrt{10})(4\sqrt{2} + 5\sqrt{10}).$
9. $(\sqrt{3} - \sqrt{7})(\sqrt{7} + \sqrt{3}).$
10. $(4\sqrt{7} - 3\sqrt{10})(4\sqrt{7} + 3\sqrt{10}).$
11. $(2\sqrt{x} - \sqrt{3x})^2.$
12. $(4\sqrt{2x - 5})^2.$
13. $(\sqrt{2} - \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5}).$

14. $\left(M + \frac{M}{2}\sqrt{5}\right)\left(M - \frac{2M}{3}\sqrt{5}\right)$.

15. $\left(\frac{P}{2}\right)^2 + \left(\frac{P}{2} - \frac{P}{2}\sqrt{2}\right)^2$.

16. $R^2 + \left(\frac{R}{2}\sqrt{3} - \frac{R}{2}\right)^2$.

Square:

17. $\sqrt{3} - \sqrt{x+2}$.

18. $\sqrt{a-2} + \sqrt{2-a}$.

19. $\sqrt{x+2} - \sqrt{2x-5}$.

20. $2\sqrt{x+1} - 5\sqrt{5x}$.

Perform the indicated multiplications:

21. $(m + \sqrt{n+p})(\sqrt{n+p} - m)$.

22. $(\sqrt{x+y} - \sqrt{x})(\sqrt{x+y} + \sqrt{x})$.

23. $(\sqrt{3y+2z} - 2\sqrt{z})(\sqrt{3y+2z} + 2\sqrt{z})$.

Express as radicals of the same order:

24. $\sqrt{3}$ and $\sqrt[3]{2}$.

25. $\sqrt[4]{2}$ and $\sqrt[5]{6}$.

26. $\sqrt[3]{3}$ and $\sqrt{8}$.

27. $\sqrt[3]{25}$ and $\sqrt[4]{9}$.

Multiply the following:

28. $\sqrt{2}, \sqrt[3]{2}$.

29. $\sqrt{x}, \sqrt[3]{a}$.

30. $\sqrt{27}, \sqrt[3]{27}$.

31. $\sqrt[4]{x^3}, \sqrt{x}$.

32. $\sqrt[4]{10}, \sqrt{2}$.

33. $\sqrt[3]{x}, \sqrt{y}$.

34. $\sqrt[3]{x^2}, \sqrt{x^3}$.

35. $\sqrt{xy}, \sqrt[3]{x^2y}$.

36. $\sqrt[4]{3m^2}, \sqrt[3]{2m^5}$.

37. $\sqrt{a-x}, \sqrt[3]{x+a}$.

192. Division of radicals. Direct division of radicals coefficient by coefficient and radicand by radicand is often possible.

Thus $$12\sqrt{2} \div 3\sqrt{5} = 4\sqrt{\tfrac{2}{5}} = \tfrac{4}{5}\sqrt{10},$$
and $$5\sqrt{st} \div 4\sqrt{t} = \tfrac{5}{4}\sqrt{s}.$$

Direct division of radicals when the divisor is a radical expression with more than one term is usually very difficult. In such cases a rationalizing factor of the denominator is used as described in the following section. We then carry out the operation of division indirectly by resorting to multiplication.

193. Rationalizing factor. One radical expression is a **rationalizing factor** for another if the product of the two is rational.

A rationalizing factor for $\sqrt{5}$ is $\sqrt{5}$, since $\sqrt{5} \cdot \sqrt{5} = 5$. For $\sqrt[3]{2}$ a rationalizing factor is $\sqrt[3]{4}$, since $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$. Similarly, $\sqrt{5} - \sqrt{3}$ is a rationalizing factor of $\sqrt{5} + \sqrt{3}$, as their product, $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$, is equal to $5 - 3$, or 2.

In like manner $(2\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - 3\sqrt{2}) = 20 - 18 = 2$.

Two important radical expressions are $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$. Two such binomials are called **conjugate** radicals, and either is a rationalizing factor for the other.

Rationalizing factors are used in division of radicals as follows:

EXAMPLES

1. $$\sqrt{3} \div \sqrt{10} = \frac{\sqrt{3}\sqrt{10}}{\sqrt{10}\sqrt{10}} = \frac{\sqrt{30}}{10}.$$

2. $$(3\sqrt{5} + 4\sqrt{7}) \div 4\sqrt{15} = \frac{(3\sqrt{5} + 4\sqrt{7}) \cdot \sqrt{15}}{4\sqrt{15} \cdot \sqrt{15}}$$
$$= \frac{15\sqrt{3} + 4\sqrt{105}}{60}.$$

3. $$(\sqrt{2}-\sqrt{3})\div(\sqrt{3}+\sqrt{5})=\frac{(\sqrt{2}-\sqrt{3})(\sqrt{3}-\sqrt{5})}{(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})}$$
$$=\frac{\sqrt{6}-3-\sqrt{10}+\sqrt{15}}{3-5}$$
$$=-\tfrac{1}{2}(\sqrt{6}-\sqrt{10}+\sqrt{15}-3).$$

Therefore when direct division of radicals is impossible, use the

Rule. *Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the fraction by a rationalizing factor for the denominator and simplify the resulting fraction.*

This rule applies in all cases, while the rule for direct division fails when dividing a real radical by an imaginary number, as will be seen in section 202.

EXERCISES

Find a simple rationalizing factor for:

1. $\sqrt{2}$.
2. $2\sqrt{5}$.
3. $\sqrt{11}$.
4. $4\sqrt{3}$.
5. $\sqrt[3]{4}$.
6. $\sqrt[4]{27}$.
7. $\sqrt{2}-\sqrt{5}$.
8. $\sqrt{6}-1$.
9. $2\sqrt{5}-3\sqrt{7}$.
10. $\sqrt{2a}-\sqrt{3x}$.
11. $\sqrt{2-a}-\sqrt{3}$.
12. $\sqrt{2-a}-3\sqrt{2a}$.
13. $\sqrt{x-y}+\sqrt{x+y}$.

Perform the indicated division:

14. $\sqrt{27}\div\sqrt{3}$.
15. $5\sqrt{6}\div\sqrt{3}$.
16. $\sqrt{15}\div\sqrt{30}$.
17. $\sqrt{m^3x}\div\sqrt{mx^3}$.
18. $(\sqrt{12}-\sqrt{8})\div\sqrt{2}$.
19. $(4\sqrt{5}-10\sqrt{75})\div(3\sqrt{5})$.
20. $(\sqrt{x^2y}+\sqrt{xy^2})\div\sqrt{xy}$.

21. $12 \div 6\sqrt{3}$. **22.** $15 \div 3\sqrt{5}$. **23.** $\sqrt{5} \div \sqrt[3]{3}$.

Solution. $\dfrac{\sqrt{5}}{\sqrt[3]{3}} = \dfrac{\sqrt{5} \cdot \sqrt[3]{9}}{\sqrt[3]{3} \cdot \sqrt[3]{9}} = \dfrac{\sqrt[6]{125 \cdot 81}}{3} = \dfrac{\sqrt[6]{10125}}{3}$

24. $\sqrt{3} \div \sqrt[3]{4}$.

25. $\sqrt{x} \div \sqrt[4]{3}$.

26. $\sqrt[4]{a} \div \sqrt{5}$.

27. $\sqrt[3]{21} \div \sqrt[5]{m^3}$.

28. $\sqrt{50} \div \sqrt[3]{2}$.

29. $\sqrt[3]{\frac{1}{9}} \div \sqrt{\frac{1}{3}}$.

30. $\sqrt{2} \div (\sqrt{2} - 3)$.

31. $\sqrt{7} \div (\sqrt{7} + \sqrt{8})$.

32. $(3\sqrt{2} + \sqrt{7}) \div (\sqrt{2} - \sqrt{7})$.

Change to respectively equivalent fractions having rational denominators:

33. $\dfrac{\sqrt{2}}{\sqrt{3} - \sqrt{7}}$. **34.** $\dfrac{\sqrt{10}}{\sqrt{2} + \sqrt{5}}$. **35.** $\dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{3} - \sqrt{2}}$.

36. $\dfrac{6\sqrt{2} + 5\sqrt{7}}{5\sqrt{2} - 6\sqrt{7}}$. **38.** $\dfrac{\sqrt{a+b} + \sqrt{c}}{\sqrt{a+b} - \sqrt{c}}$.

37. $\dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$. **39.** $\dfrac{9}{\sqrt{3 - \sqrt{3}}}$.

40. By the use of the table of square roots, calculate the expressions in Exercises 36 and 39 in the form given and in the required form with rational denominators.

Rationalize the denominators of:

41. $\dfrac{5}{\sqrt[4]{2} + 1}$. **42.** $\dfrac{7}{\sqrt[4]{3} - 2}$.

Perform the indicated division:

43. $(\sqrt{11} + \sqrt{6}) \div (\sqrt{11} - \sqrt{6})$.

44. $(\sqrt{a+x} - \sqrt{z}) \div (\sqrt{a+x} + \sqrt{z})$.

45. $(\sqrt{10}-\sqrt{7}-\sqrt{2})\div(\sqrt{7}+\sqrt{2})$.

46. Is there any difference in meaning between the direction before Ex. 33 and that before Ex. 43?

47. Does $3+\sqrt{13}$ satisfy $x^2-6x-4=0$?

48. Does $\dfrac{\sqrt{2}-2}{3}$ satisfy $3x^2-4x+2=0$?

49. Does $\frac{1}{7}(1\pm\sqrt{22})$ satisfy $7x^2-2x-3=0$?

194. Square root of surd expressions. The square of a binomial is usually a trinomial. However, the result of squaring a binomial of the form $\sqrt{a}+\sqrt{b}$ is a binomial if a and b are rational numbers. Thus

$$(\sqrt{5}-\sqrt{2})^2=5-2\sqrt{10}+2=7-2\sqrt{10}.$$

In $7-2\sqrt{10}$, 7 is the sum of 5 and 2, and 10 is the product of 5 and 2. These relations and the fact that the coefficient of the radical $\sqrt{10}$ is 2 enable us to find the square root of many expressions of the form $a\pm\sqrt{b}$ by writing in the form of $x\pm2\sqrt{xy}+y$ and then extracting the square root of the trinomial square as follows.

EXAMPLE

Extract the square roots of $8-\sqrt{60}$.

Solution. $8-\sqrt{60}=8-2\sqrt{15}$.

We must now find two numbers whose sum is 8 and whose product is 15. These are 3 and 5.

Then $8-2\sqrt{15}=5-2\sqrt{15}+3=(\sqrt{5}-\sqrt{3})^2$.

Hence the square roots of $8-\sqrt{60}$ are $\pm(\sqrt{5}-\sqrt{3})$.

EXERCISES

Find the positive square roots in Exercises 1–10.

1. $7 - 2\sqrt{6}$.
2. $7 - 2\sqrt{12}$.
3. $16 - \sqrt{32}$.
4. $17 + 2\sqrt{16}$.
5. $21 + 14\sqrt{2}$.
6. $13 + \sqrt{160}$.
7. $22 + \sqrt{228}$.
8. $14\,x + \sqrt{96\,x^2}$.
9. $2\,p - 2\sqrt{p^2 - 1}$.
10. $6 + 2\sqrt{9 - a^2}$.
11. $\sqrt{9+4\sqrt{5}} = \sqrt{?} + \sqrt{?}$.
12. $\sqrt{13 + 2\sqrt{30}} = ?$
13. $\sqrt{x + \sqrt{x^2 - 4\,b^2}} = ?$
14. $\sqrt{a^3 + a^2 + 3\,a + 2\,a\sqrt{3\,a^2 + 3\,a}} = ?$

NOTE. In the writings of one of the later Hindu mathematicians (about A.D. 1150) we find a method of extracting the square root of surds which is practically the same as that given in the text. In fact, the formula for the operation is given, apart from the modern symbols, as follows: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$. The study of expressions of the type $\sqrt{\sqrt{a} \pm \sqrt{b}}$ had been carried to a most remarkable degree of accuracy by the Greek, Euclid. His researches on this subject, if original with him, place him among the keenest mathematicians of all time, but his work and all of his results are expressed in geometric language which is very far removed from the algebraic symbolism of to-day.

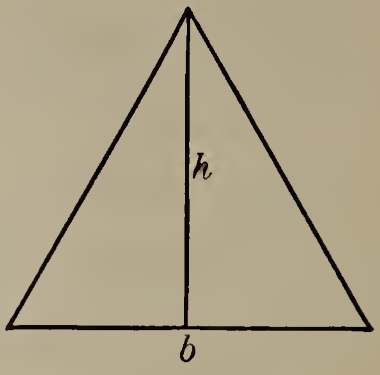
PROBLEMS

(Obtain answers in simplest radical form and correct to the nearest second decimal place, where the results are numeric and involve decimals.)

1. The side of an equilateral triangle is 5 inches. Find the altitude and the area.

2. The altitude of an equilateral triangle is 36. Find one side and the area.

3. The side of an equilateral triangle is b. Find the altitude and the area.

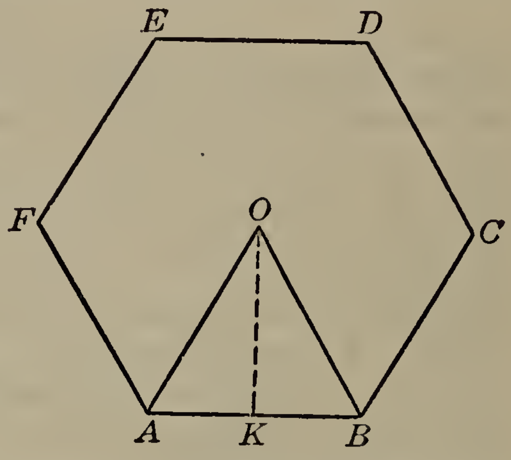


4. The area of an equilateral triangle is A. What is the altitude and one side?

5. A triangle has sides 4 inches, 8 inches, and 10 inches. Find the area of the triangle. Find the altitude on the side of 8 inches.

HINT. Let the altitude on the side 8 be x, and the two parts into which the altitude divides side 8 be y and $8 - y$; then set up the equations involving x and y, and solve.

6. Find the altitude on the shortest side of the triangle whose sides are 9 inches, 5 inches, and 12 inches.



Fact from Geometry. A regular hexagon may be divided into six equal equilateral triangles by lines from its center to the vertices.

In the adjacent regular hexagon, $AB = BC = CD$, etc. O is the center and OK perpendicular to AB is the **apothem** of the hexagon.

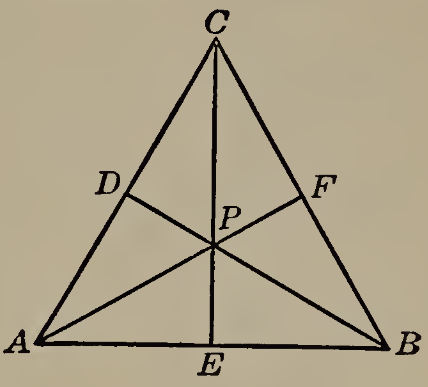
7. Find the apothem and the area of a regular hexagon (*a*) whose side is 20, (*b*) whose side is m.

8. Find the side and the area of a regular hexagon (*a*) whose apothem is 25, (*b*) whose apothem is a.

Facts from Geometry. The volume of a pyramid or cone is $\frac{ab}{3}$, where a is the altitude and b is the area of the base.

The altitudes of an equilateral triangle intersect at a point which divides each altitude into two parts whose ratio is 2 to 1.

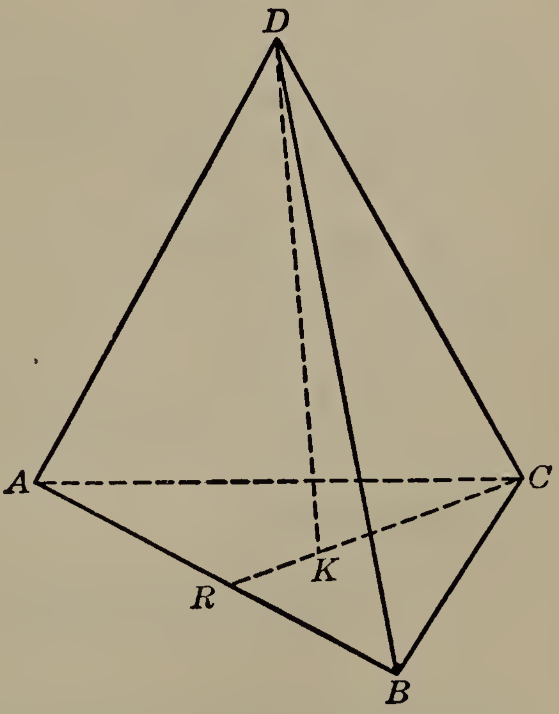
9. The sides of a regular pyramid with a square base are equilateral triangles of sides 32 feet. Find the altitude and the volume of the pyramid.



10. The side of an equilateral triangle is 40 inches. Find the two parts into which each altitude is divided by the other altitudes. What is the area of each of the six smaller triangles thus formed?

A **regular tetrahedron** is a pyramid whose four faces are equal equilateral triangles.

The altitude of a regular tetrahedron (*DK* in the adjacent figure) meets the base at the point where the altitudes of the base intersect.



11. *ABCD* is a regular tetrahedron. If each edge is 36, find *CR*, *CK*, and the altitude *DK*.

12. Find the altitude and volume of a regular tetrahedron each of whose edges is 25 inches.

13. Show that the altitude and volume of a regular tetrahedron whose edge is e are

$$\frac{e}{3}\sqrt{6} \text{ and } \frac{e^3}{12}\sqrt{2} \text{ respectively.}$$

MISCELLANEOUS EXERCISES

Reduce to equivalent fractions having rational denominators:

1. $\frac{2\sqrt{3}}{\sqrt{2}}$.

2. $\frac{3}{7\sqrt{5}}$.

3. $\frac{4}{\sqrt{2}-\sqrt{5}}$.

4. $\frac{6}{6^{\frac{1}{2}}-5^{\frac{1}{2}}}$.

5. $\frac{3^{\frac{1}{2}}+7^{\frac{1}{2}}}{2\cdot 3^{\frac{1}{2}}-3\cdot 7^{\frac{1}{2}}}$.

6. $\frac{3\sqrt{7}+2\sqrt{3}}{3\sqrt{7}-2\sqrt{3}}$.

7. $\frac{x\sqrt{y}-a\sqrt{z}}{x\sqrt{y}+a\sqrt{z}}$.

8. $\frac{d-\sqrt{d^2-4}}{d+\sqrt{d^2-4}}$.

9. $\frac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.

10. $\frac{x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}$.

Find the positive square roots of the following:

11. $25+2\sqrt{66}$.

12. $49-14\sqrt{6}$.

13. $16+8\sqrt{3}$.

14. $25-10\sqrt{6}$.

15. Show that

$$\frac{2+\sqrt{7}}{3} \text{ is a root of } 3x^2-4x-1=0.$$

16. Show that

$$\frac{2\pm 5\sqrt{3}}{4} \text{ are roots of } 16x^2-16x=71.$$

Simplify, using the table at the end of the book; in securing results correct to two decimal places:

17. $3\sqrt{50}-5\sqrt{8}+\sqrt{2}-\frac{1}{7}\sqrt{98}$.

18. $4\sqrt{\frac{2}{3}}-2\sqrt{\frac{1}{6}}+21\sqrt{\frac{3}{2}}+\sqrt{150}$.

19. $\sqrt{10} \div \sqrt{20}$.

20. $\sqrt{7} \div 5\sqrt{3}$.

21. $(8\sqrt{6} - \sqrt{7}) \div (\sqrt{7} + \sqrt{2})$.

Simplify:

22. $\sqrt{15} + 5\sqrt{\frac{1}{15}} - \sqrt{\frac{3}{5}} + 3\sqrt{\frac{5}{3}}$.

23. $y \cdot \frac{1}{5}(x^2 - y^2)^{-\frac{1}{2}}(-10y) - (x^2 - y^2)^{\frac{1}{2}}$.

24. $\left(\dfrac{m^5 \cdot n \cdot m^{-1} - n \cdot 5m^4}{(m^4)^3}\right) \div \dfrac{n}{m^3}$.

25. $\dfrac{\sqrt{m}}{\sqrt{m} + \sqrt{n}} - \dfrac{\sqrt{n}}{\sqrt{m} - \sqrt{n}}$.

26. $\dfrac{(i^a + i^{-a})(i^a + i^{-a}) - (i^a - i^{-a})(i^a - i^{-a})}{(i^a + i^{-a})}$.

27. $$\frac{\dfrac{(m^3 - 1)m^{(b-1)}b + m^b \cdot 3m}{(m^3 - 1)^2}}{\dfrac{m^b}{m^3 - 1}}.$$

28. $$\frac{\dfrac{y^3(ay^{a-2}) - (y^a + 1)2y}{(y^3)^2}}{\dfrac{y^a + 1}{y^3}}.$$

29. $$\frac{\dfrac{m^{3a}bm^{b-1} - m^b \cdot 3am^{2a-1}}{(m^{2a})^3}}{\dfrac{m^b}{m^{3a}}}.$$

30. $$\frac{\dfrac{2z^{-3}(-3z^{-1}) - (z^3 + 2)(-6z^{-6})}{(z^{-4})^3}}{\dfrac{z^{-3} + 2}{z^{-4}}}.$$

CHAPTER XXVII

QUADRATIC EQUATIONS

(In Part Review)

195. Solution by completing the square. The quadratic equation is defined on page 169. The method of solving a quadratic equation by completing the square is explained and illustrated on pages 326–328.

EXERCISES

Solve by completing the square and check real results as directed by the teacher:

1. $x^2 + 6x - 7 = 0.$

2. $x^2 - 4x - 6 = 0.$

3. $x^2 - 28 = 3x.$

4. $x^2 - 2x = 11.$

5. $2x^2 - 2x - 5 = 0.$

6. $6x^2 + 6 + 13x = 0.$

7. $6x^2 + 13x + 2 = 0.$

8. $3x^2 - 14x = 5.$

9. $3x^2 - 10x + 5 = 9x - 10 - 3x^2.$

10. $x^2 = \dfrac{x + 15}{2}.$

11. $(2m + 1)^2 - (m - 1)^2 = -2.$

In Exercises 12–18 obtain results to the nearest hundredth:

12. $x^2 - 4x + 2 = 0.$

HINTS. By applying the rule we get

$$x = 2 + \sqrt{2}.$$
$$x = 2 - \sqrt{2}.$$

From the table $\sqrt{2} = 1.414.$

This gives us $x = 3.41$ and $.59.$

13. $x^2 - 2x - 2 = 0.$

14. $p^2 - 12p + 29 = 0.$

15. $m^2 + 8m + 10 = 0.$

16. $2x^2 - 2 - x\sqrt{2} = 0.$

17. $3x^2 + 2x\sqrt{2} - 2 = 0.$

18. $\frac{x^2}{12} - \frac{x}{3} = \frac{3}{4}.$

19. $x^4 - 5x^2 + 6 = 0.$

NOTE. This is not a quadratic equation, but many equations of this form can be solved by the methods applicable to quadratics.

Solution.

$$\begin{aligned} x^4 - 5x^2 + 6 &= 0. \\ x^4 - 5x^2 &= -6. \\ x^4 - 5x^2 + \tfrac{25}{4} &= \tfrac{25}{4} - \tfrac{24}{4} = \tfrac{1}{4}. \\ (x^2 - \tfrac{5}{2})^2 &= \tfrac{1}{4}. \\ x^2 - \tfrac{5}{2} &= \pm \tfrac{1}{2}. \\ x^2 &= 2 \text{ and } 3. \end{aligned}$$

Whence $x = \pm\sqrt{2}$ and $\pm\sqrt{3}$.

Check as usual.

NOTE. It should be particularly observed that the equation of Exercise 19 has four roots instead of two. In general an equation has a number of roots equal to its degree. Thus the equation in Exercise 22 has six roots, although some of them are imaginary. Consequently the student at present cannot expect to find them all.

20. $x^4 - 6x^2 + 5 = 0.$

21. $x^4 - 4x^2 - 45 = 0.$

22. $x^6 + x^3 - 12 = 0.$

23. $x^6 - 2x^3 = 15.$

24. $2x^4 - 7x^2 + 6 = 0.$

25. $x^4 - 6x^2 + 8 = 0.$

26. $y^6 - 8y^3 + 12 = 0.$

27. $4(x^2 - x)^2 - 19(x^2 - x) = -22.$

Solution.

Let $x^2 - x = y.$

Substituting y for $x^2 - x$, we get

$$4y^2 - 19y = -22.$$

Solving, $y = 2$ and $\frac{11}{4}.$

Then $x^2 - x = 2.$

Whence $x = -1 \text{ and } 2.$

Also $x^2 - x = \frac{11}{4}.$

Whence $x = \frac{1}{2} \pm \sqrt{3}.$

In Exercises 28–30 do not expand but solve as in Exercise 27:

28. $(x + 2)^2 + 3(x + 2) - 4 = 0.$

29. $(y^2 - 3y)^2 - 13(y^2 - 3y) = -30.$

30. $2(m^2 - 2m)^2 - 13(m^2 - 2m) + 15 = 0.$

31. $ax^2 + bx + c = 0.$

Solution. $ax^2 + bx + c = 0.$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

32. $2x^2 + ax - 6a = 0.$

33. $cz^2 - cz = 3 - 3z.$

34. $cw^2 + 3aw = c - 3a.$

35. $abx^2 + x(a^2 - b^2) = ab.$

36. $2x^4 + a^2x^2 = 6a^4.$

37. $6m^2 + bmx - 2b^2x^2 = 0.$

38. $(3pz + 4m)^2 = (2pz - m)^2.$

39. $2b = ax\sqrt{b} + a^2x^2$.

40. $(2x + c)^2 + 2x + c = 2$.

41. $\dfrac{a^2}{y^2} = \dfrac{a-3}{y-3}$.

42. $\dfrac{x^2}{3x-1} = \dfrac{n^2}{3n-1}$.

43. $\dfrac{n^2+2n+4}{x^2+2x+4} = \dfrac{n^2}{x^2}$.

44. $\dfrac{x^2+3}{a^2+3} = \dfrac{x}{a}$.

196. Solution by formula. In Exercise 31, above, the general quadratic $ax^2 + bx + c = 0$ has been solved and the roots found to be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (F)$$

The expression (F) is a general result and may be used as a formula to solve any quadratic equation which is in the standard form $ax^2 + bx + c = 0$, where a, b, and c may represent numbers, single letters, binomials, or any other form of algebraic expression not involving x.

If the numbers a, b, and c are such that the expression $b^2 - 4ac$ is negative, the formula contains the square root of a negative number, which is a kind of number not yet fully considered in this text. In the exercises that follow it will be assumed that only such numeric values of the literal coefficients are involved as will not make the radicand $b^2 - 4ac$ negative. A discussion of the case here ruled out will be found in Chapter XXIX.

EXERCISES

Solve for x by formula and check results as directed by the teacher:

1. $2x^2 + 5x = 1$.

Solution. Writing in standard form,

$$2x^2 + 5x - 1 = 0.$$

Comparing with $ax^2 + bx + c = 0$, we see that 2 corresponds to a, 5 to b, and -1 to c. Substituting these values in the formula (F) gives

$$x = \frac{-5 \pm \sqrt{25 - 4 \cdot 2 \cdot (-1)}}{4}$$

$$= \frac{-5 \pm \sqrt{33}}{4}.$$

Check as usual.

Solve by formula:

2. $x^2 - 2x - 15 = 0.$

3. $x^2 + x - 1 = 0.$

4. $x^2 + 7x + 1 = 0.$

5. $x = x^2 - 1.$

6. $x^2 - 5x - 5 = 0.$

7. $4x^2 - 8x - 1 = 0.$

8. $3x^2 + 6x + 2 = 0.$

9. $2x^2 - 4x + 1 = 0.$

10. $9x^2 - 42x + 38 = 0.$

11. $2k^2x^2 - kx - 6 = 0.$

Solution. $2k^2x^2 - kx - 6 = 0.$

Here $a = 2k^2$, $b = -k$, and $c = -6$.

Substituting these values in formula (F),

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4 \cdot 2k^2(-6)}}{2 \cdot 2k^2}$$

$$= \frac{k \pm \sqrt{k^2 + 48k^2}}{4k^2} = \frac{k \pm 7k}{4k^2} = \frac{2}{k} \text{ and } -\frac{3}{2k}.$$

Check as usual.

12. $x^2 - 3bx - 4b^2 = 0.$

13. $6x^2 + 7px - 18p^2 = 0.$

14. $6p^2x^2 - 7px + 2 = 0.$

15. $q(x^2 + p^2) = pq(q^2 + 1).$

16. $x + 2\sqrt{a} = \frac{3a}{x}.$

17. $16\,x^2 + rx = s.$

18. $2\,x^2 + ac = (2\,c + a)x.$

19. $x^2 + px + 2 - px^2 + 3\,x = 0.$

Solution. $(1 - p)x^2 + (p - 3)x + 2 = 0.$

Here $a = 1 - p,\ b = p - 3,$ and $c = 2.$

Hence
$$x = \frac{3 - p \pm \sqrt{(p - 3)^2 - 8(1 - p)}}{2(1 - p)}$$
$$= \frac{3 - p \pm \sqrt{p^2 + 2\,p + 1}}{2(1 - p)}.$$
$$x = \frac{3 - p \pm (p + 1)}{2 - 2\,p}$$
$$= \frac{2}{1 - p} \text{ and } 1.$$

Check as usual.

20. $x^2 + a^2 + x - a = 2\,ax.$

21. $x^2 + x - 2 + k(x^2 + 2\,x) = 0.$

22. $a^2x^2 - 2\,ax + 1 = ax^2 - x.$

23. $b^2x^2 + 2 = x^2 + 3\,bx + x.$

24. $x^2 - 6\,x + 3\,xy - 12\,y + 8 = 0.$

25. $2\,x^2 + x + 2\,xy + 3\,y = 3.$

26. $x^2 - 3\,x - y^2 - 3\,y = 0.$

27. $x^2 - 5\,xy - x + 6\,y^2 - 2 - y = 0.$

197. Comparison of the various methods. Three methods have been given for the solution of the quadratic equation:

(*a*) Solution by factoring (p. 169).
(*b*) Solution by completing the square (p. 326).
(*c*) Solution by formula (p. 441).

In practice, the first and the last of these methods are most convenient. A graphical method for the solution of a quadratic equation is presented on pages 469–477.

When an equation with integral coefficients can be solved by factoring, as explained in Chapter XV, the roots are always rational numbers.

For example, $10x^2 - 11x - 6 = 0$ factors into $(2x - 3)(5x + 2) = 0$. Hence the roots are $x = \frac{3}{2}$ and $-\frac{2}{5}$.

An inspection of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

shows that, if a, b, and c are integers, one always obtains roots involving radicals unless the expression under the radical sign, $b^2 - 4ac$, is a perfect square. In this case, however, the values of the roots are rational numbers.

For example, in the equation $10x^2 - 11x - 6 = 0$ the value of the expression $b^2 - 4ac$ is $(-11)^2 - 4 \cdot 10(-6) = 361$, which is a perfect square. Hence the roots of $10x^2 - 11x - 6 = 0$ are rational numbers, since the radical term in the roots can be expressed as a rational number.

Hence to determine whether a quadratic equation of the form $ax^2 + bx + c = 0$ has rational roots, we have the

RULE. *Compute the value of* $b^2 - 4ac$ *for the equation.*

If the result is the square of an integer, the left member of the equation can be factored and the roots are rational.

ORAL EXERCISES

Determine which of the following have rational roots:

1. $x^2 + 3x + 2 = 0$.
2. $x^2 - 4x + 10 = 0$.
3. $x^2 - 4x - 4 = 0$.
4. $2x^2 + 5x + 2 = 0$.

5. $2x^2 + 5x + 3 = 0.$

6. $x^2 - 2x - 1 = 0.$

7. $4x^2 - x - 3 = 0.$

8. $2x^2 - x + 1 = 0.$

9. $2x^2 - 6x - 9 = 0.$

10. $3x^2 - 5x - 12 = 0.$

11. $9x^2 - 9x - 3 = 0.$

12. $10x^2 + 6x + 1 = 0.$

REVIEW EXERCISES

Solve the following by the method best adapted to each:

1. $x^2 - 8x + 12 = 0.$

2. $3x^2 - 2x - 5 = 0.$

3. $5x^2 + 7x + 1 = 0.$

4. $4x^3 = x.$

5. $x + 2 = 6x^2.$

6. $x^3 - 2x^2 - x + 2 = 0.$

7. $x^2 - x - 1 = 0.$

8. $x^2 - 5.5x + 7.36 = 0.$

9. $3(x - 3)^2 - 11(x - 3) - 4 = 0.$

10. $(x^2 - 1)^2 - 4(x^2 - 1) - 5 = 0.$

11. $.09x^2 - .21x + .1 = 0.$

12. $x^2 - x + .24 = 0.$

13. $3x^2 - 12.3x + 7.8 = 0.$

14. $\dfrac{x-1}{x} = \dfrac{x-1}{6}.$

15. $\dfrac{3x-2}{x-1} - \dfrac{2x+1}{x+1} = 0.$

16. $\dfrac{x-7}{2x-10} = \dfrac{x+5}{3x+21}.$

17. $\dfrac{x}{2x-.3} = \dfrac{3.4}{x}.$

18. $x^2 - 8ax + 16a^2 - 9b^2 = 0.$

19. $2x^2 + 6x - 4a^2 = x^2 - 9.$

20. $10x^2 + 11x + .028 = 0.$

21. $\dfrac{x^2 - 2.8}{x} = \dfrac{3.1}{2}.$

22. $\dfrac{(x-2)^3-(x+1)^3}{x}=\dfrac{2x-31}{2}.$

23. $\dfrac{x^3-6}{x}=3-2x.$

24. $5(2x+1)(x-5)=(4x-3)(5x+29).$

25. $2(3-4x)(2x+1)=(6x-1)(2x-11).$

26. $\dfrac{x^2}{3x-5}=\dfrac{m^2}{3m-5}.$

27. $\dfrac{x^2}{c^2}=\dfrac{x^2+x+1}{c^2+c+1}.$

28. $\dfrac{x^2}{m^2}=\dfrac{ax+b}{am+b}.$

29. $(3x+4)^2-(2x+3)^2=0.$

30. $39abx=10a^2+14b^2x^2.$

31. $9x^2-2k^2+2km+3kx+3mx=0.$

32. $59x^2=15x^4+52.$

33. $7.3x^2-11.1x-6.3=0.$

34. $.51x^2+.73x+.16=0.$

35. $4+\dfrac{a}{a-2x}=\dfrac{a+2x}{a}.$

36. $(4x+7)(x+2)-(x+3)(2x+5)=0.$

37. $a^2x^2+1=b^2x^2+2ax.$

38. $\dfrac{.3x-2}{.1x-1}+\dfrac{.5x+5}{.4x-1}=8\tfrac{1}{2}.$

39. $x^2-3a=ax-3x.$

40. $\dfrac{x}{\sqrt{3}}=\dfrac{2\sqrt{5}}{x}+3.$

41. $\dfrac{a}{x+a}+\dfrac{b}{x+b}-\dfrac{2c}{x+c}=0.$

42. $\dfrac{a+x}{b+x}+\dfrac{b+x}{a+x}=\dfrac{5}{2}.$

43. $\dfrac{ax^2+bx+c}{bx^2+mx+n}=\dfrac{c}{n}.$

44. $\dfrac{2x+a}{ax+2}=\dfrac{a+x}{2+x}.$

45. $\dfrac{ax+b}{bx+a}=\dfrac{mx-n}{nx-m}.$

PROBLEMS

1. Separate 56 into two parts such that one part shall be the square of the other.

2. Separate 45 into two parts such that one part shall be twice the square of the other.

3. Find two consecutive even numbers whose product is 168.

4. Find two integers whose difference is 4 and whose product is 117.

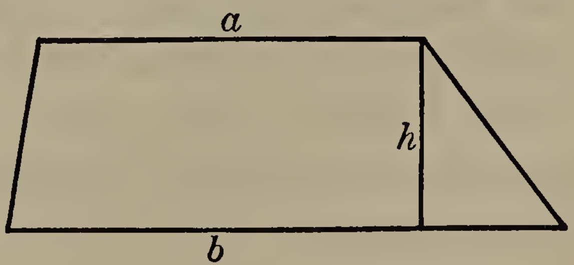
5. The product of two consecutive numbers is one greater than the sum of the integers next smaller and next greater than these numbers. Find the two numbers.

6. One side of a rectangle is 6 feet longer than the other. The area of the rectangle is 91 square feet. Find the sides.

7. The altitude of a triangle is 4 feet more than its base. The area is 16 square feet. Find the altitude and the base.

8. One base of a trapezoid is twice as long as the altitude, while the other base is 17 feet longer than the altitude. The area of the trapezoid is 45 square feet. Find the lengths of the bases.

HINT. The area of a trapezoid is given by the formula $A=\dfrac{h(a+b)}{2}$, where h is the altitude and a and b are the bases.

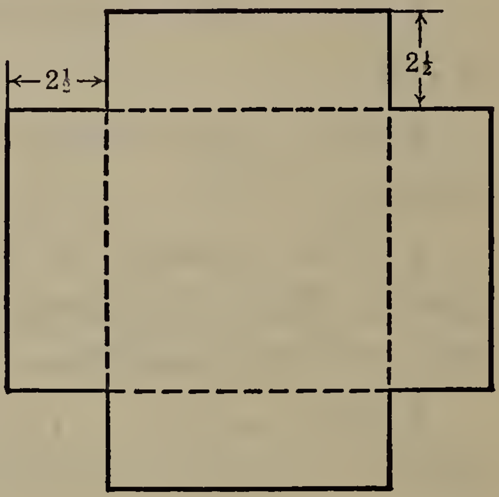


9. A polygon of n sides always has $\frac{1}{2}n(n-3)$ diagonals. How many sides has a polygon with 135 diagonals?

10. The side of one square is 8 inches greater than that of a second square. The sum of the areas of the two squares is 424 square inches. Find the side of each square.

11. The radius of one circle is 2 inches greater than the radius of a second circle. The sum of their areas is $106\frac{6}{7}$ square inches. Find the radius of each circle.

12. A $2\frac{1}{2}$ inch square is cut from each corner of a square piece of tin. The sides are then turned up to form an open box of volume 80 cubic inches. What is the side of the original square?



13. A tinsmith wishes to make a tin box 4 inches deep which will hold 180 cubic inches, by cutting squares from the corners of a square of tin and folding up the sides. How large a square of tin will he require?

14. What positive value of x will make the product of $x-3$ and $x+3$ greater by 10 than their difference?

15. What values of x will make the product of $3x-2$ and $x+3$ equal to $7x+2$?

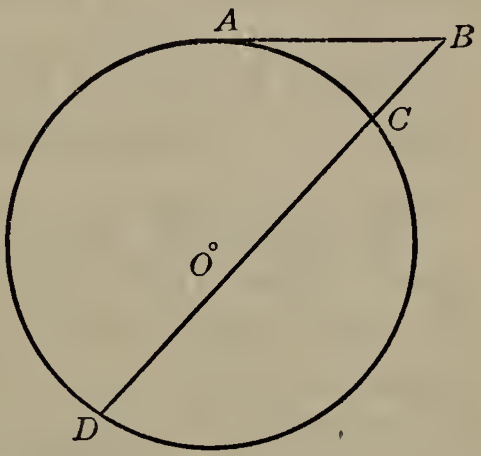
16. The length of a room is 4 feet greater than its width. A rug placed in the middle of the room leaves a margin 2 feet wide on each side. If the entire margin is just equal to the area of the rug, find the dimensions of the room.

17. A lawn is 36 by 24 yards. How wide a strip must be cut around it with a lawn mower in order to leave just $\frac{5}{8}$ of it uncut?

18. A farmer has fencing enough to go around a lot whose length is 3 rods more than twice its width. If he uses the same amount of fencing to inclose a square lot, the area inclosed will be increased by 36 square rods. How many rods of fencing has he?

19. The side of a cabin is six feet high, and is to be built of boards of uniform width. If the boards had each been one inch wider it would have required one less board to cover the side of the cabin. How wide were the boards?

20. If AB in the accompanying figure is a tangent to the circle, and BD is any secant, then $(AB)^2 = BC \cdot BD$. If $AB = 6$ and $CD = 9$, find BC.



21. The distance from the horizon to the top of a cliff is 20 miles. How high is the cliff? (Earth's radius = 4000 miles.)

22. The sum of \$2000 is invested, and at the end of the first year the year's interest plus \$900 is added to the investment. At the end of the second year the investment amounts to \$3100. What is the rate of interest?

23. The distance in feet, s, through which a body falls from rest in t seconds, neglecting air resistance, is $s = 16\,t^2$. A bomb is dropped from an aëroplane and strikes the ground 12 seconds later. How high is the plane?

24. If a projectile is thrown vertically into the air with a velocity of 200 feet per second, its distance above the ground after t seconds is $s = 200\,t - 16\,t^2$ feet, neglecting air-resistance. After how many seconds will it be 600 feet above the ground? After how many seconds will the projectile return to the ground? Explain the zero root.

CHAPTER XXVIII

IRRATIONAL EQUATIONS

198. Definitions and discussion. An irrational equation in one unknown is an equation in which the unknown occurs under a radical, or is affected by a fractional exponent.

Thus $\sqrt{x} - 2x + 1 = 0$, and $x^{\frac{1}{4}} - x^{\frac{1}{2}} + 1 = 0$, and $y - (2y)^{\frac{1}{2}} - 4 = 0$ are irrational equations.

One difficulty involved in the solution of such equations arises from the fact that sometimes results are obtained which do not satisfy the given equation and hence are not roots of that equation. A result of this kind is called extraneous.

EXAMPLE

	(*a*) Solve $\sqrt{x-3} - 5 = 0$.	(*b*) Solve $-\sqrt{x-3} - 5 = 0$.
Solution. Transposing,		
	$\sqrt{x-3} = 5.$ (1)	$-\sqrt{x-3} = 5.$ (1)
Squaring,	$x - 3 = 25.$ (2)	$x - 3 = 25.$ (2)
Solving,	$x = 28.$	$x = 28.$
Check.	$\sqrt{28-3} - 5 = 0.$	$-\sqrt{28-3} - 5 = 0.$
	$\sqrt{25} - 5 = 0.$	$-\sqrt{25} - 5 = 0.$
	$5 - 5 = 0,$	$-5 - 5 = 0,$
	which is true	which is not true.

It appears from a study of these solutions that statements (1) differ only in the signs preceding their left members. Consequently this distinction disappears after squaring, and equations (2) are identical. Since the remainder of the work in both (*a*) and (*b*) consists in the solution of (2), the result obtained is really the root of this equation. Whether the root obtained satisfies both (*a*) and (*b*), or only one of them, can be determined only by substitution. In this case it appears that (*a*) is an equation and that (*b*) is not, but is merely a false statement in the form of an equation.

In any case, all of the roots of the original equation are sure to be among the results found, provided no factor containing the unknown has been divided out. But **no result should be called a root unless it satisfies the original equation.** This means that all results must be checked.

In irrational equations, as in all the work up to the present, it is understood that unless a radical or an expression affected by a fractional exponent is preceded by the double sign ± it has only the one value, just like any other number symbol.

Thus $\sqrt{81}$ means $+9$, and not -9.
Also $9^{\frac{1}{2}}$ means $+3$, while $-9^{\frac{1}{2}}$ means $-\sqrt{9}$, or -3.
and $x^{\frac{1}{2}}$ means $+\sqrt{x}$ and not $-\sqrt{x}$.

If this fact is kept in mind, it is clear from an inspection of (*b*), above, that it could have no root, since the sum of two negative numbers could not possibly be zero.

The method of solving equations in which an unknown occurs under a radical is stated in the

RULE. *Transpose the terms so that one radical expression (the least simple one if there are two or more) is the only term in one member of the equation.*

Next raise both members of the resulting equation to the same power as the index of this radical.

Combine like terms in each member, and, if radical expressions still remain, repeat the two preceding operations until an equation is obtained which is free from radicals; then solve this equation.

CHECK. *Substitute in the original equation the values found and reduce the resulting numeric equation to its simplest form by extracting roots, but not by raising both members of the equation to any power.*

Finally, reject all extraneous roots.

EXERCISES

Solve the following for real roots and check results, rejecting all extraneous roots:

1. $\sqrt{x+2}=3.$

2. $\sqrt{5x+11}=6.$

3. $\sqrt{3x+7}=10.$

4. $3\sqrt{2x+3}-2=7.$

5. $\sqrt[3]{4x+4}=2.$

6. $2\sqrt[3]{3x+1}+5=7.$

7. $x+\sqrt{x+8}+2=0.$

Solution.

$$\begin{aligned} \sqrt{x+8} &= -(x+2).\\ x+8 &= x^2+4x+4.\\ x^2+3x-4 &= 0;\\ x &= -4, \text{ or } 1. \end{aligned}$$

Check.

Substituting -4 for x,

$$-4+\sqrt{4}+2=-2+2=0,$$

which is true.

Substituting 1 for x,

$$\begin{aligned} 1+\sqrt{9}+2 &= 0\\ 6 &= 0, \end{aligned}$$

which is not true.

Hence 1 is extraneous.

8. $x - \sqrt{x} - 2 = 0.$ **10.** $2(x-4)^{\frac{1}{2}} = (2x+3)^{\frac{1}{2}}.$

9. $\sqrt{x-3} = \sqrt{4x-5}.$ **11.** $x\sqrt{6} = \sqrt{x+2}.$

12. $\sqrt{2}(x+1)^{\frac{1}{2}} = \sqrt{3}(5x+2)^{\frac{1}{2}}.$

13. $\sqrt{x^2+4x-5} = \sqrt{2-2x}.$

14. $x = 11 - 3\sqrt{x+7}.$ **15.** $5 - \sqrt{2x+5} = \sqrt{2x}.$

16. $\sqrt{x+1} = \sqrt{x^2+3x+2}.$

17. $\sqrt{x+1} = \sqrt[4]{x+1}.$

18. $(3x-4)^{\frac{1}{2}} + (4x+3)^{\frac{1}{2}} = 0.$

19. $\sqrt{x-2} = \sqrt[4]{x^2-6x+1}.$

20. $\sqrt{2x+5} = 1 - \sqrt{x+2}.$

Solution. $\sqrt{2x+5} = 1 - \sqrt{x+2}.$

Squaring, $2x+5 = 1 - 2\sqrt{x+2} + x + 2.$

Transposing and collecting,

$$x + 2 = -2\sqrt{x+2}.$$

Squaring,

$$x^2 + 4x + 4 = 4x + 8,$$

or

$$x = \pm 2.$$

Check. Substituting $+2$ for x in the original equation,

$$\sqrt{9} = 1 - \sqrt{4} \text{ or } 3 = -1,$$

which is not true.

Substituting -2 for x,

$$\sqrt{1} = 1 - \sqrt{0}, \text{ or } 1 = 1.$$

Hence -2 is the only root.

21. $\sqrt{x-2} = \sqrt{x} + 2.$

22. $\sqrt{2x+3} + 3 = 3\sqrt{x+1}.$

23. $\sqrt{x-1}+\sqrt{x}=\sqrt{2x-1}$.

24. $\sqrt{x-1}+\sqrt{3x+1}-2=0$.

25. $\sqrt{3(x+2)}+\sqrt{x+4}=\sqrt{7x+1}$.

26. $x-4=\dfrac{(x+4)^{\frac{1}{2}}}{3}$.

27. $\sqrt{x+3}+\sqrt{x}=\sqrt{3}$.

28. $1+\sqrt{x+2}=\sqrt{x}$.

29. $\sqrt{x+2}-\sqrt{3x+4}=\sqrt{2x+2}$.

30. $x^{\frac{4}{3}}-10x^{\frac{2}{3}}+9=0$.

NOTE. The equation here given is not a quadratic equation, but it is of the general type $ax^{2n}+bx^n+c=0$. Here x occurs in but two terms, and its exponent in one term is twice that in the other term. Many equations in this form can be solved by completing the square (compare Exercise 19, p. 439).

Solution. $$x^{\frac{4}{3}}-10x^{\frac{2}{3}}+9=0.$$

$$(x^{\frac{2}{3}}-9)(x^{\frac{2}{3}}-1)=0.$$

Hence $$x^{\frac{2}{3}}=9 \text{ and } 1.$$

Thus $$x^{\frac{1}{3}}=\pm 3 \text{ and } x=\pm 27$$

or $$x^{\frac{1}{3}}=\pm 1 \text{ and } x=\pm 1.$$

Check. Substituting ± 27 for x in the original equation,

$$(\pm 27)^{\frac{4}{3}}-10(\pm 27)^{\frac{2}{3}}+9=0,$$

or $$81-10\cdot 9+9=0.$$

Substituting ± 1 for x in the original equation,

$$(\pm 1)^{\frac{4}{3}}-10(\pm 1)^{\frac{2}{3}}+9=0$$

or $$1-10+9=0.$$

31. $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} - 8 = 0.$

32. $x^5 + 4x^{\frac{5}{2}} - 5 = 0.$

33. $x = x^{\frac{1}{2}} + 2.$

34. $x^{-2} + 5x^{-1} + 6 = 0.$

35. $\frac{3}{x} - 10 = x^{-\frac{1}{2}}.$

36. $6x^3 + 7x^{\frac{3}{2}} + 1 = 0.$

37. $8x^3 - 7x^{\frac{3}{2}} - 1 = 0.$

38. $8x^3 + 7x^{\frac{3}{2}} - 1 = 0.$

39. $x^{\frac{1}{3}} + x^{\frac{1}{6}} - 2 = 0.$

40. $\sqrt[4]{x} = \sqrt{x} - 2.$

41. $\sqrt{3 - 2x} = x + 30.$

42. $6x^2 - \frac{165}{x^2} = 3.$

43. $3x - 11x^{\frac{1}{2}} - 20 = 0.$

44. $\sqrt{2x - 1} + \sqrt{x + 4} = 6.$

45. $\sqrt{3x + 1} - \sqrt{2x - 1} = 1.$

46. $\sqrt{3x - 2} + \sqrt{2x + 5} = 5.$

47. $\sqrt{3x + 4} - \sqrt{2x - 4} = 2.$

48. $\frac{3\sqrt{x + 4}}{\sqrt{3x} - \sqrt{6}} = \sqrt{6}.$

49. $\frac{\sqrt{x} - 3}{\sqrt{x}} - \frac{5 - \sqrt{x}}{4} = 0.$

50. $\frac{\sqrt{x} - 1}{\sqrt{x} + 1} = \frac{\sqrt{2}}{\sqrt{3}}.$

51. $\frac{2 - \sqrt{x}}{\sqrt{2x}} = \frac{3\sqrt{x} - 2}{\sqrt{2}}.$

52. $\frac{x^{\frac{1}{2}} - 5}{2 - 2x^{\frac{1}{2}}} = \frac{3x^{\frac{1}{2}}}{7}.$

53. $\frac{2a^{\frac{1}{2}}}{(2x - a)^{\frac{1}{2}}} = \frac{(2x + 4a)^{\frac{1}{2}}}{3a^{\frac{1}{2}}}.$

54. $\frac{(x - 4)^{\frac{1}{2}}}{(2x + 3)^{\frac{1}{2}}} + \frac{(2x + 3)^{\frac{1}{2}}}{(x - 4)^{\frac{1}{2}}} = 2.$

CHAPTER XXIX

IMAGINARIES

199. Introduction. As we have progressed in the study of algebra, nearly every forward step has involved the use of a more complicated and refined type of number. In arithmetic the positive integer and the positive fraction were sufficient for all requirements. In the beginning of algebra the negative integer and fraction were introduced. The solution of quadratic equations such as, $x^2 - 2 = 0$, forced upon us the irrational number, and led to the study of methods of operating with radicals, and to the solution of radical equations. Each new kind of number has presented itself as the root of an equation that we were supposed to solve, and with the introduction of each new number the power and generality of our algebraic method was increased.

Up to the present the square root of a negative number has been avoided, or dismissed with the remark that it is an imaginary. It is true that we cannot imagine any length that the number $\sqrt{-2}$ could measure. The positive numbers are all that we need for measurement. Similarly the person with very simple mathematical needs might remark, that since positive integers are adequate for counting, the fractions and irrational numbers are unnecessary.

Just as we have defined and used the operations of addition, subtraction, multiplication, and division with negative numbers (Chapter III) and with irrational numbers (Chapter XXI), so now we will define the meaning of these operations on the so-called imaginaries. The introduction of these numbers enables us to solve completely the quadratic equation for all cases. They are frequently used in many branches of applied mathematics, especially in the theory of electricity.

The equation $x^2 + 1 = 0$, or $x^2 = -1$, states that x is a number whose square is -1. By defining a new number, $\sqrt{-1}$, as one whose square is -1, we obtain one root for the equation $x^2 + 1 = 0$.

Similarly, $\sqrt{-5}$ is a number whose square is -5. And, in general, $\sqrt{-n}$ is a number whose square is $-n$. Obviously, $\sqrt{-5}$ means something very different from $\sqrt{5}$, and $\sqrt{-n}$ from $\sqrt{n}$.

The positive numbers are all multiples of the unit $+1$, and the negative numbers are all multiples of the unit -1. Similarly, pure imaginary numbers are real multiples of the imaginary unit $\sqrt{-1}$, as $2\sqrt{-1}$, $-5\sqrt{-1}$, and $b\sqrt{-1}$.

Furthermore, $\sqrt{-4} = \sqrt{4 \cdot (-1)} = 2\sqrt{-1}$; $\sqrt{-a^2} = \sqrt{a^2(-1)} = a\sqrt{-1}$; and $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1}$.

The imaginary unit $\sqrt{-1}$ is often denoted by the letter i; that is, $3\sqrt{-1} = 3i$.

If a real number be united to a pure imaginary by a plus sign or a minus sign, the expression thus obtained is called a **complex number**.

Thus $-2 + \sqrt{-1}$ and $3 - 2\sqrt{-4}$ are complex numbers. The general form of a complex number is $a + bi$, in which a and b may be any real numbers.

NOTE. Up to the time of Gauss (1777–1855) complex numbers were not clearly understood and were usually thought of as absurd.

The situation reminds one of the time when negative numbers were similarly regarded, and the veil was removed from both in about the same way. It was found that negative numbers really had a significance — that they could be used in problems that involve debt, opposite directions, and many other everyday relations. The interpretation of imaginary numbers is not quite so obvious, and is not considered in this text. But as soon as it was seen that an interpretation was possible the ice was broken, and it needed only the insight and authority of a man like Gauss to give complex numbers their proper place in mathematics.

ORAL EXERCISES

Express as multiples of $\sqrt{-1}$ or i:

1. $\sqrt{-9}$.
2. $\sqrt{-36}$.
3. $\sqrt{-x^2}$.
4. $3\sqrt{-2}$.
5. $2\sqrt{-5}$.
6. $5\sqrt{-b}$.
7. $\sqrt{6}\cdot\sqrt{-6}$.
8. $\sqrt{3}\cdot\sqrt{-2}$.
9. $\sqrt{7}\cdot\sqrt{-28}$.
10. $x\sqrt{-y}$.
11. $\sqrt{-a^2-4a-4}$.
12. $\sqrt{-x^2-12x-36}$.

200. Addition and subtraction of imaginaries. The fundamental operations of addition and subtraction are performed with imaginary and complex numbers as they are performed with rational numbers and ordinary radicals of similar form.

Thus $$3\sqrt{-1}+2\sqrt{-1}=5\sqrt{-1},$$

and $$7\sqrt{-1}-\sqrt{-16}=7\sqrt{-1}-4\sqrt{-1}=3\sqrt{-1},$$

and $$3(-1)^{\frac{1}{2}}+5(-1)^{\frac{1}{2}}-4\sqrt{-1}=4i.$$

Also $$(2+3\sqrt{-1})+(5-2\sqrt{-1})=7+\sqrt{-1}.$$

Similarly $$(x+yi)+(m+ni)=m+x+(n+y)i.$$

EXERCISES

Simplify and express results as multiples of i:

1. $4\sqrt{-1} + 3\sqrt{-1}.$

2. $3\sqrt{-1} + \sqrt{-4}.$

3. $\sqrt{-16} - \sqrt{-9}.$

4. $\sqrt{-25} + \sqrt{-36}.$

5. $\sqrt{-9} + \sqrt{-4}.$

6. $(-27)^{\frac{1}{2}} + (-18)^{\frac{1}{2}}.$

7. $\sqrt{-8} + \sqrt{-32}.$

8. $2\sqrt{-36\,a^2} - 5\sqrt{-49\,a^2}.$

9. $3 + 2\sqrt{-4} + 5 - 6\sqrt{-1}.$

10. $3\sqrt{-x^2} + 2\,x - 5\sqrt{-x^2}.$

11. $(2\,m + 3\,ni) + (m - 2\,ni).$

12. $16 - 9\,i + 5 - 3\sqrt{-25}.$

13. $7 - 4\sqrt{-16\,a^2} - 5\sqrt{-36\,a^2} + 2\sqrt{-4\,a^2} + 6.$

14. $30 - 2(-1)^{\frac{1}{2}} + 7(-16)^{\frac{1}{2}} + 5.$

15. $3\sqrt{-3} + 2\sqrt{-8} - 3\sqrt{-32} + 4\sqrt{-48}.$

16. $(9 - 3\sqrt{-25}) + (8 + 2\sqrt{-9}).$

17. $\sqrt{-16\,a^6} + 3\,a^3\sqrt{-4} + 2\sqrt{-12} + 5\sqrt{-27}.$

18. $(2\,a + 3\,bi) - (5\,a - 6\,bi).$

201. Multiplication of imaginaries. By the definition of the square root of a number, the square of $(-n)^{\frac{1}{2}}$ is $-n$.

Therefore $(\sqrt{-1})^2 = -1,$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -\sqrt{-1},$$

and

$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = 1.$$

To multiply $\sqrt{-3}$ by $\sqrt{-5}$, we first write

$$\sqrt{-3} = \sqrt{3} \cdot \sqrt{-1}$$

and

$$\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1}.$$

Then $$\sqrt{-3}\cdot\sqrt{-5}=(\sqrt{3}\cdot\sqrt{-1})(\sqrt{5}\cdot\sqrt{-1})$$
$$=\sqrt{15}\cdot\sqrt{-1}\cdot\sqrt{-1}=\sqrt{15}(-1)$$
$$=-\sqrt{15}.$$

Similarly
$$(4\sqrt{-2})(-\sqrt{3}\sqrt{-7})=(4\sqrt{2}\cdot\sqrt{-1})(-\sqrt{3}\sqrt{7}\sqrt{-1})$$
$$=-4\sqrt{42}(-1)=+4\sqrt{42}.$$

In general, if $\sqrt{-a}$ and $\sqrt{-b}$ are two imaginaries whose product is desired, they should first be written in the form $\sqrt{a}\cdot\sqrt{-1}$ and $\sqrt{b}\cdot\sqrt{-1}$ or $\sqrt{a}\,i$ and $\sqrt{b}\,i$ and the multiplication should not be performed until this has been done. This method will prevent many errors.

In this connection it must be clearly understood that one rule followed in the multiplication of real radicals (see page 313) does not apply to imaginary numbers.

In the case of ordinary radicals we have
$$\sqrt{2}\cdot\sqrt{3}=\sqrt{2\cdot 3}=\sqrt{6}.$$

But the product of two imaginaries like $\sqrt{-2}\cdot\sqrt{-3}$ does not equal $\sqrt{(-2)(-3)}$, for this equals $\sqrt{6}$. We have seen above that $\sqrt{-2}\cdot\sqrt{-3}=\sqrt{2}\cdot\sqrt{3}(\sqrt{-1})^2=-\sqrt{6}$.

In multiplying two complex numbers, write each expression in the form $a \pm bi$ and proceed as in the following

EXAMPLE

Multiply $3+\sqrt{-2}$ by $5-\sqrt{-5}$.

Solution. $$3+\sqrt{-2}=3+\sqrt{2}\sqrt{-1} \qquad (1)$$
and $$5-\sqrt{-5}=5-\sqrt{5}\sqrt{-1}. \qquad (2)$$

Multiplying (1) by (2),
$$15+5\sqrt{2}\sqrt{-1}-3\sqrt{5}\sqrt{-1}-\sqrt{10}(-1).$$

Rewriting
$$15+\sqrt{10}+(5\sqrt{2}-3\sqrt{5})\sqrt{-1}.$$

EXERCISES

Perform the following indicated multiplications and simplify results:

1. $(-1)^4$.

2. $(-1)^5$.

3. $(-1)^6$.

4. $(-1)^7$.

5. $(\sqrt{-1})^6$.

6. $(\sqrt{-1})^5$.

7. $(\sqrt{-2})^7$.

8. $(\sqrt{-3})^3$.

9. $(3\sqrt{-7})^5$.

10. $\sqrt{-9} \cdot \sqrt{-16}$.

11. $\sqrt{-25}\,(-\sqrt{-4})$.

12. $\sqrt{-5}\,(-\sqrt{-3})$.

13. $\sqrt{-9} \cdot \sqrt{-15}$.

14. $3\sqrt{-5} \cdot 5\sqrt{-3}$.

15. $\sqrt{-x} \cdot \sqrt{-y}$.

16. $5\sqrt{-10}(-3\sqrt{-7})$.

17. $\sqrt{a+b} \cdot \sqrt{-a-b}$.

18. $(2+\sqrt{-1})(2-\sqrt{-1})$.

19. $(4+\sqrt{-2})(4-\sqrt{-2})$.

20. $(\sqrt{5}+\sqrt{-3})(\sqrt{5}-\sqrt{-3})$.

21. $(2+\sqrt{-4})(4-\sqrt{-2})$.

22. $(7+\sqrt{-3})(5+\sqrt{-6})$.

23. $(2-6\,i\sqrt{3})(2+3\,i\sqrt{3})$.

24. $(5+\sqrt{-1})(3-\sqrt{-1})$.

25. $(6+2\,i)(3-2\,i\sqrt{3})$.

26. $(m+ni)(p+qi)$.

27. $(x+bi)^2$.

28. $(m+xi)(m-xi)$.

29. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-2})^2$.

30. $(-\frac{1}{2}-\frac{1}{2}\sqrt{2})^2$.

31. $(-3+3\sqrt{-5})^3$.

32. $(-3-3\sqrt{-5})^3$.

33. $(m-ni)^3$.

34. $(x+yi)^2-(x-yi)^2$.

35. $(2+2\sqrt{-5})^3-(2-2\sqrt{-5})^3$.

36. $(m+i\sqrt{1-x^2})(m-i\sqrt{1-x^2})$.

37. Determine whether (*a*) the sum, and (*b*) the product, of the numbers $3+4\sqrt{-2}$ and $3-4\sqrt{-2}$ are real.

38. Determine whether (*a*) the sum, and (*b*) the product, of the numbers $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are real.

202. Division of imaginaries. One complex number is called the *conjugate* of another if their product and their sum are real. Thus $a + bi$ and $a - bi$ are conjugates. Conjugate complex numbers are used in division of imaginary expressions as conjugate radicals are used in division of real radicals.

In case either the numerator or the denominator of a fraction is imaginary or complex, the division may be performed as in the following

EXAMPLES

1. $\sqrt{-10} \div \sqrt{5}$.

Solution.
$$\frac{\sqrt{-10}}{\sqrt{5}} = \frac{\sqrt{5}\ \ \sqrt{-10}}{(\sqrt{5})^2} = \frac{5\sqrt{-2}}{5}$$
$$= \sqrt{2}\,i.$$

2. $\sqrt{27} \div \sqrt{-3}$.

Solution.
$$\frac{\sqrt{27}}{\sqrt{-3}} = \frac{\sqrt{27} \cdot \sqrt{-3}}{(\sqrt{-3})^2} = \frac{9\,i}{-3}$$
$$= -3\,i.$$

3. $\sqrt{-15} \div \sqrt{-5}$.

Solution.
$$\frac{\sqrt{-15}}{\sqrt{-5}} = \frac{\sqrt{15}\,i}{\sqrt{5}\,i} = \sqrt{3}.$$

4. $2 \div (3+\sqrt{-5})$.

Solution.
$$\frac{2}{3+\sqrt{-5}} = \frac{2}{3+\sqrt{5}\,i} \cdot \frac{3-\sqrt{5}\,i}{3-\sqrt{5}\,i}$$
$$= \frac{6-2\sqrt{5}\,i}{9+5}$$
$$= \frac{3-\sqrt{5}\,i}{7}.$$

The method of the above examples is stated in the

Rule. *Write the dividend over the divisor in the form of a fraction.*

Then multiply both numerator and denominator of this fraction by the simplest expression which will make the new denominator real and rational.

Reduce the result to its simplest form.

EXERCISES

Perform the indicated operations:

1. $\sqrt{-6} \div \sqrt{2}$.

2. $\sqrt{15} \div \sqrt{-3}$.

3. $6\sqrt{3} \div 3\sqrt{-1}$.

4. $\sqrt{-8} \div \sqrt{-2}$.

5. $2 \div \sqrt{-2}$.

6. $1 \div \sqrt{-7}$.

7. $\sqrt{27} \div \sqrt{-3}$.

8. $(-25)^{\frac{1}{2}} \div (-81)^{\frac{1}{2}}$.

9. $\sqrt{ab} \div \sqrt{-a}$.

10. $\sqrt{-x} \div \sqrt{-y}$.

11. $(-10\,am)^{\frac{1}{2}} \div (-3\,m)^{\frac{1}{2}}$.

12. $[(-b^4)^{\frac{1}{2}} - (-b^8)^{\frac{1}{2}}] \div (-b)^{\frac{1}{2}}$.

13. $5 \div (1 - \sqrt{-1})$.

14. $3 \div (2 + \sqrt{-3})$.

15. $3\sqrt{-2} \div (\sqrt{-1} - 3)$.

16. $4\sqrt{-3} \div (2\sqrt{-5} + 3)$.

17. $(-2 + 3\sqrt{-2}) \div (-2 - 3\sqrt{-2})$.

18. $(1 - 3\,i) \div (5 + 7\,i)$.

19. $a \div (a + bi)$.

20. $(x + iy) \div (m + in)$.

21. $(2 + 3\,i)(i - 1) \div (2 - 5\,i)(i + 1)$.

22. Is $\frac{3}{2}(1 + \sqrt{-3})$ a cube root of -27?

23. Does $x^2 - 4x + 16 = 0$ if $x = 2 + 2\sqrt{-3}$?

24. Does $-1 + 2\sqrt{-1}$ satisfy $x^2 + 2x + 5 = 0$?

25. Does $x = -\frac{2}{3}\sqrt{-2}$, $y = \frac{1}{3}\sqrt{-2}$ satisfy the system

$$9x^2 - 18xy - 27y^2 = -10,$$
$$3x^2 - 6xy - 15y^2 = -2?$$

203. Equations with imaginary roots. The student should now be able to solve and to check the solution of an equation which has imaginary roots.

EXERCISES

Solve the equations which follow, and check the results:

1. $x^2 + 2x + 4 = 0.$
2. $x^2 - 6x + 12 = 0.$
3. $x^2 + 5x + 9 = 0.$
4. $x^2 - 3x + 15 = 0.$
5. $2x^2 + 3x + 4 = 0.$
6. $x^2 + x + 1 = 0.$
7. $x^2 - x + 1 = 0.$
8. $6x^2 - 8x + 15 = 0.$
9. $7x^2 + 12x + 8 = 0.$
10. $3x^2 + 8x + 10 = 0.$
11. $x^3 = 1.$

HINT. If $x^3 = 1,$

then $x^3 - 1 = 0,$

and $(x - 1)(x^2 + x + 1) = 0,$

or $x - 1 = 0,$

and $x^2 + x + 1 = 0.$

12. $x^3 = -1.$
13. $x^3 = 27.$
14. $x^3 = -8.$
15. $x^4 = 1.$
16. $x^4 = 25.$
17. $x^6 = 1.$
18. $x^6 = 64.$
19. $x^3 = 125.$
20. $x^3 = -64.$

21. How many square roots has any real number? cube roots? fourth roots? sixth roots?

22. What do the preceding exercises suggest regarding the number of nth roots which any number has?

23. $8x^3 - 1 = 0.$

24. $27x^3 + 64 = 0.$

25. $x^4 - 4x^2 - 21 = 0.$

26. $x^4 - x^3 - 3x^2 + 3x = 0.$

27. $x^4 + 5x^2 - 6 = 0.$

28. $3x^4 + 10x^2 + 3 = 0.$

29. $6x^4 - 7x^2 - 3 = 0.$

30. $2x^4 + 5x^2 + 2 = 0.$

31. $15x^4 + 17x^2 + 4 = 0.$

32. $(x^2 + 9)(x^2 + 5x + 8) = 0.$

33. $(x^2 + 3x)^2 - 2(x^2 + 3x) - 8 = 0.$

34. $(2x^2 + 3x)^2 - 41(2x^2 + 3x) + 378 = 0.$

NOTE. Long before the time of Gauss, mathematicians had performed the operations of multiplication and division on complex numbers by the same rules that they used for real numbers. As early as 1545 Cardan showed that the product of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ was the real number 40. However, he was not always equally fortunate in obtaining correct results, for in another place he sets $\frac{1}{4}(-\sqrt{-\frac{1}{4}}) = \frac{1}{\sqrt{64}} = \frac{1}{8}.$

Even the rather complicated formula for extracting any root of a complex number was discovered in the early part of the eighteenth century. But all these operations were purely formal, and seemed to most mathematicians a mere juggling with symbols until Gauss showed clearly the place and usefulness of such numbers.

NOTE ON THE USE OF IMAGINARIES. We have explained the laws of addition, subtraction, multiplication, and division for imaginary (and complex) numbers and have made some use of them. It is largely because imaginaries obey these laws that we call them numbers, for it must be admitted that we cannot count objects with imaginary numbers. Nor can we state by means of them our age, our weight, or the area of the earth's surface. It should be remembered, however, that we can do none of these things with negative numbers. We may have a group of objects — books, for example — whose number is 5; but no group of *objects* exists whose number is -5, or -3, or any negative number whatever. If it is asked, How, then, can negative numbers and imaginary numbers have any practical use? the answer is this: They have a practical use because when they enter into our calculations and we have performed the necessary operations upon them and obtained our final result, that result can frequently be interpreted as a concrete number like those dealt with in ordinary arithmetic. Moreover, if the result cannot be so interpreted, it is, in applied mathematics at least, finally rejected.

In that part of electrical engineering where the theory and measurement of alternating currents are treated, complex numbers have had extensive use. Their use in the difficult problems which there arise has given a briefer, a more direct, and a more general treatment than where such numbers are not used.

In theoretical mathematics complex numbers have been of great value in many ways. For example, numerous important theorems about functions are more easily proved under the assumption that the variable is complex. Then, by letting the imaginary part of the complex number become zero, we obtain the proof of the theorem for real values of the variable. Indeed, the student need not go very far beyond this point in his mathematical work to learn that, if e is 2.7182^+ (see page 528), $e^{\sqrt{-1}} + e^{-\sqrt{-1}}$ is equal to the real number 1.082^+. At the same time he will learn also how such a form arises, and something of its importance. In a way which we cannot now explain, even so involved an expression as $(a + ib)^{c+id}$ has in higher work a meaning and a use. If the student pursues his mathematical studies far enough, that meaning and use and a multitude of other uses for complex numbers will become familiar to him. But the numbers which we have learned in this book to use, namely fractions, negative numbers, irrational numbers, and complex numbers, complete the number system of ordinary algebra, for it can be proved that from the fundamental operations no other forms of number can arise.

CHAPTER XXX

FUNCTIONS AND THEIR GRAPHS

204. Functions. The notion of function is one of the central concepts in modern mathematics. The basic idea involved is that of the dependence of one quantity on one or more other quantities. Countless functional relations exist in our everyday affairs, most of which are too complicated for mathematical expression. For example, the height of a boy is a function of his age, his health, his heredity, and many other elements. But it would be impossible to measure exactly the effect of health or heredity on the height of a boy. Hence this function is not one that can be dealt with mathematically. On the other hand, the distance that a body falls from a state of rest under the influence of the force of gravity can be measured and expressed by the formula $s = \frac{1}{2} gt^2$, in which s equals the distance, g equals 32, and t equals the elapsed time in seconds. In fact the formula can be derived theoretically from certain laws of nature. Consequently, this function is one that can be used mathematically. In general one variable is a function of another if it depends on the other for its value. Thus, in $x = y + 5$, x is a function of y or vice versa.

In the study of algebra we are concerned with the mathematical formulas in terms of which the physical relations are expressed, rather than with the actual phenomena themselves. Consequently in this chapter we shall content ourselves with a study of the graphical representation of equations that actually express many of the relationships which we know in our everyday life.

205. Names of functions. A function is called **linear, quadratic,** or **cubic** according as its degree with respect to the unknown or unknowns is first, second, or third respectively.

Thus $3x - 5$ is a linear function of x; $2x^2 - 5x + 8$ is a quadratic function of x; and $y^3 - 2y^2 + y - 10$ is a cubic function of y.

In the study of functions the unknown is often called the variable, since from this point of view the problem is not so much the finding of an unknown as it is the study of the changes of a variable quantity.

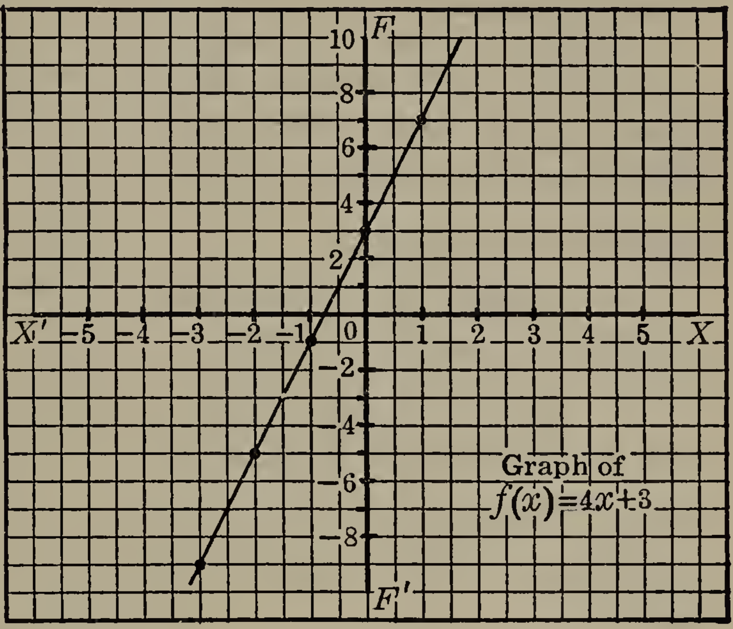
206. Notations for a function. After a function of any variable x has once been given it is usual to refer to it later in the same discussion by the symbol $f(x)$, which is read **the function of x**, or more briefly, **f of x**.

207. Linear functions. The expression $4x + 3$ is a function of x, and the value of this binomial varies with x. The following table gives a partial view of the relative change of values between x and the function $4x + 3$:

If $x =$	-4	-3	-2	-1	0	1	2	3	4
then $f(x) = 4x + 3 =$	-13	-9	-5	-1	3	7	11	15	19

This relation can be represented graphically by using the same x-axis as before (section 167) and using the y-axis as the function axis; that is, laying off values of x horizontally and corresponding values of the function $4x + 3$

vertically. The graph resulting from the above table of values is shown in the accompanying figure. It can be shown that the graph of a linear function is always a straight line.



EXERCISES

1. Construct the graph of the function $2x + 3$.

2. Construct the graph of the function $5x - 1$.

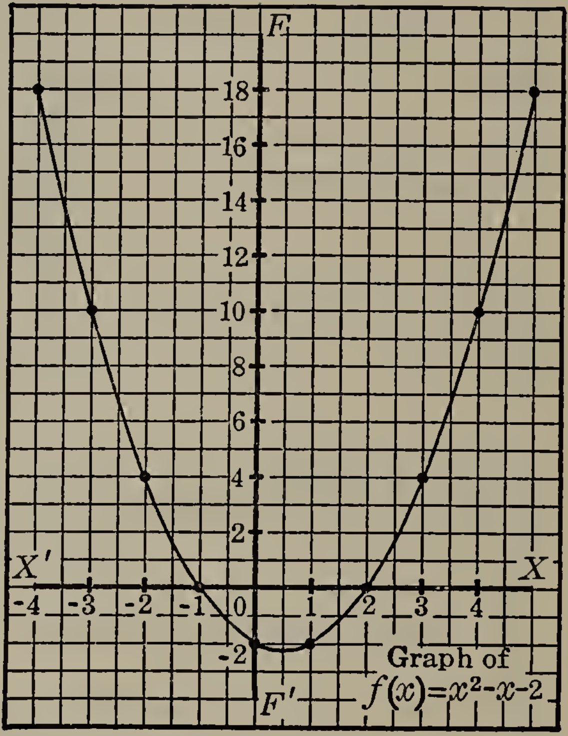
3. Construct the graph of the function $3x + 7$.

4. The relation between the readings on the Fahrenheit thermometer scale and those on the Centigrade scale is expressed by the equation $F = \frac{9}{5}C + 32$. Construct the graph for $f(C) = \frac{9}{5}C + 32$. What is the Fahrenheit reading when the temperature is 50° Centigrade? 0°? 100°?

5. An automobile starts to ascend a grade at a speed of 30 miles per hour, but each second slows up 2 miles per hour. The function of the time which gives the speed of the car is $V = f(t) = 30 - 2t$. Construct the graph of the function $f(t) = 30 - 2t$. Read from the graph the speed of the car after 5 seconds, after 15 seconds, after 20 seconds.

208. Quadratic functions. The function $f(x) = x^2 - x - 2$ may be represented graphically as follows:

If $x =$	-4	-3	-2	-1	0	1	2	3	4
then $f(x) = x^2 - x - 2 =$	18	10	4	0	-2	-2	0	4	10



Plotting the points corresponding to the numbers in the table we obtain the accompanying graph.

The graph of a quadratic function in one variable is a curve called a **parabola**. It may be sharper or flatter than the accompanying graph, but is always of the same general shape.

EXERCISES

Construct the graphs of the following:

1. $f(x) = x^2 + 2x - 5.$

2. $f(x) = 4x^2 - 3x + 1.$

3. $f(x) = 2x^2 - 2x - 3.$

4. $f(x) = x^2 - 7x + 6.$

5. $f(x) = 3x^2 + 5x - 10.$

6. $f(x) = 5x^2 - 3x + 5.$

7. A body falling from rest under the influence of gravity follows the law $s = f(t) = 16t^2$, in which s is the distance fallen in feet, and t is the time, in seconds, elapsed since the body started falling. Construct the graph of $f(t) = 16t^2$.

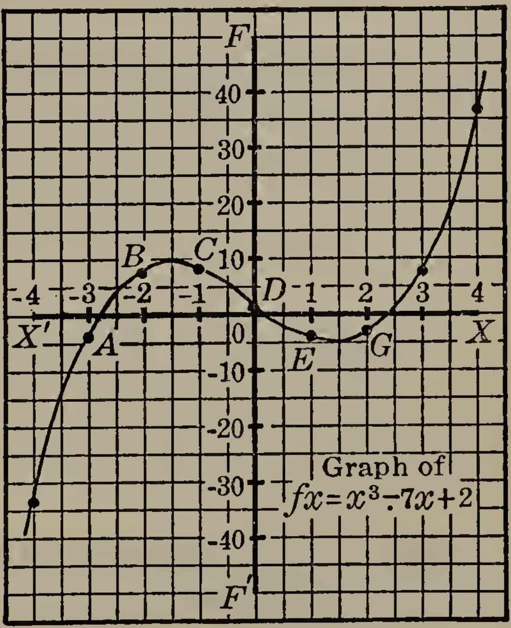
NOTE. In the study of analytical geometry one takes up systematically the curves which represent equations of the various degrees, beginning with the simplest. It turns out, as we have already seen, that the linear equation is represented by a straight line. Equations of the second degree in x and y lead to the so-called "conic sections."

One of the most interesting and important aspects of the graphical method is the fact that the simplest equations correspond to the most useful curves both in pure science and in nature. Among the commonest curves in nature are the circle and the parabola. Their equations are the very simplest equations of the second degree.

209. Graph of a cubic function. The graph of a cubic function is obtained in the same general way as that of a quadratic function. The function $x^3 - 7x + 2$ may be represented graphically by proceeding as follows:

If $x =$	-4	-3	-2	-1	0	1	2	3	4
then $f(x) = x^3 - 7x + 2 =$	-34	-4	8	9	2	-4	-4	8	38

Plotting the points corresponding to the numbers in table (as in the figure below), we obtain the points $A(-3, -4)$, B, C, D, E, and G, in the order named. The curve crosses the x-axis three times: once between 2 and 3, again between 0 and 1, and a third time between -2 and -3. Above G the curve rises indefinitely, and below A it falls indefinitely. In each case it becomes more and more nearly straight as it recedes from the x-axis, never crossing either axis again.



In forming a table of values two pairs are sufficient for a linear function, but more are needed for a quadratic function and still more for a cubic function. Usually the higher the degree of the function, the more points are needed in constructing its graph. It should be noted that in making a good graph the number of points is not so important as is their distribution, which should be such as faithfully to outline the entire curve. Where the graph curves rapidly or makes sharp turns the points should be close together. Such places are difficult to locate before the graph is constructed; hence one should make a table of values which appears to be sufficient and plot them. Then inspection of the plotted points will usually show where sharp turns or rapid curvature exists. The table of values should then be properly extended and the additional points located. Repetition of this last step will enable one to draw a graph which accurately pictures the variation of the function.

It should be observed here that scales on the two axes need not be the same. Some experience is required to choose for the two axes the scales which are best suited to bring out clearly the shape of the curve. In general the graph should be drawn to *as large* a scale, *in both directions*, as the size of the paper permits. What that will be for each axis can be decided by inspecting the table of values. For example, when the dimensions of the preceding graph are once determined one can see from the table that all values of x are easily represented but that it may be undesirable to try to represent the values of the function, less than -34 or greater than 38.

EXERCISES

Construct the graphs of the following:

1. $f(x) = x^3 + 2x - 1$.
2. $f(x) = x^3 - 5x - 3$.
3. $f(x) = x^3 - 3x + 1$.
4. $f(x) = x^4 - 12x^2 + 24$.

NOTE. The notion of a function is one of the three or four most fundamental ideas in modern mathematics. Only the simplest examples are given in this book, but many others involving expressions of the utmost complexity have been studied by mathematicians for many years. An important reason for the study of functions is found in the fact that all kinds of facts and principles which we meet in the study of nature can be expressed symbolically by means of functions, and the discovery of the properties of such functions helps us to understand the meaning of the facts. A complete understanding of the laws of falling bodies, light, electricity, or sound could never be reached without the study of the mathematical functions which these phenomena suggest.

When the electrician, the architect, or the artillerist meets a problem, he frequently must represent quantities by letters. The x and the y may represent the measures of objects in nature, but the solution has become merely an operation of algebra. As students of algebra we are not concerned with the origin of the function or expression, but merely with the numeric determination of some unknown or the simplification of some expression.

210. Graphical solution of equations in one unknown. In elementary algebra one of the most important applications of the preceding sections is their use in solving equations in one unknown. The ideas involved can be made clear by questions on the graphs of sections 207, 208, and 209.

ORAL EXERCISES

1. From the graph in section 207, determine the value of the function $f(x) = 4x + 3$ at the point where its graph crosses the x-axis.

2. Does the value of x for this point satisfy the equation $4x + 3 = 0$?

3. Solve $4x + 3 = 0$ without reference to the graph.

4. What point on the graph represents the root of $4x + 3 = 0$?

5. From the graph in section 208, determine the values of the function $f(x) = x^2 - x - 2$ at the points where its graph crosses the x-axis.

6. Do the values of x for these points satisfy the equation $x^2 - x - 2 = 0$?

7. Solve $x^2 - x - 2 = 0$ without referring to the graph.

8. What points on the graph represent the roots of $x^2 - x - 2 = 0$?

9. From the graph in section 209, determine the values of the function $x^3 - 7x + 2$ at the points where its graph crosses the x-axis.

10. Do these values of x make the function $x^3 - 7x + 2$ equal zero?

11. What method could be used to solve the equation $x^3 - 7x + 2 = 0$?

211. The process of graphical solution. From what precedes, it is apparent that the steps in the graphical solution of an equation in one unknown are:

I. *Transpose the terms so that the right member is zero.*
II. *Graph the function in the left member.*
III. *The values of* x *for the points where the graph crosses the* x*-axis are the real roots of the equation.*

The algebraic solutions of a linear equation and a quadratic equation in one unknown are so simple that except for the purpose of illustration their graphical solution is comparatively unimportant. The algebraic solution of cubic and higher equations, except in simple cases, is much more difficult than the solution of the quadratic and is never presented in an elementary course. For this reason and for the insight it gives into equations in general, the *graphical* solution of the cubic and higher equations is important and illuminating. For such equations if the method of factoring fails, the graphical method is the only method open to the student at this point in his progress.

EXERCISES

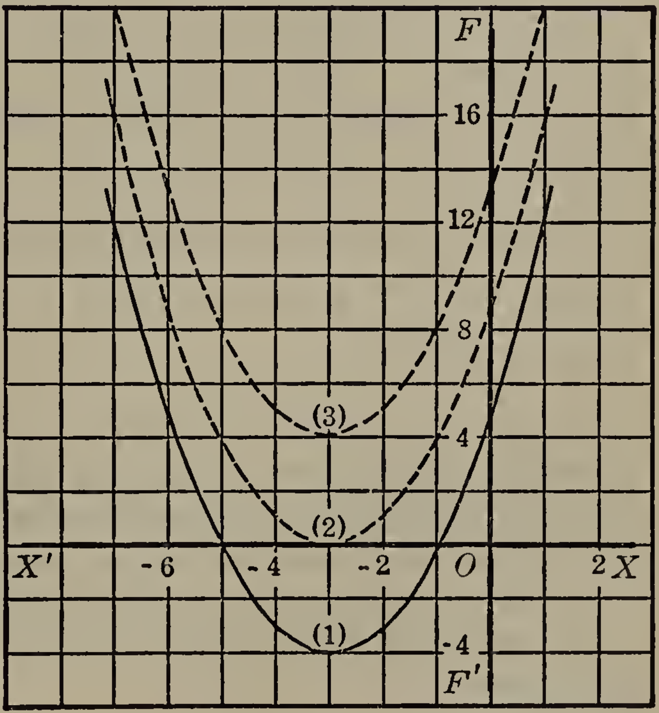
Solve graphically obtaining results to the first decimal place:

1. $x^2 + 2x - 1 = 0$.
2. $x^3 - 10x + 5 = 0$.
3. $x^3 - 8x - 10 = 0$.
4. $x^3 - 6x = 0$.
5. $x^3 - 3x^2 + 1 = 0$.
6. $x^4 - 12x^2 + 15 = 0$.

212. Imaginary roots. An equation of the second or a higher degree often has **imaginary roots.** Such roots cannot be obtained by the graphical methods so far considered. A study of the graphs which follow will make clear why this is true.

Consider the following equations:

$$x^2 + 6x + 5 = 0, \tag{1}$$
$$x^2 + 6x + 9 = 0, \tag{2}$$
$$x^2 + 6x + 13 = 0. \tag{3}$$



The graphs of the functions in the left members of equations (1), (2), and (3) are given in the accompanying figure. The three functions differ only in their constant terms, for 4 added to the constant term of (1) gives the constant term of (2), and 4 added to the constant term of (2) gives the constant term of (3). Apparently, as the constant term is increased the graph rises without change of shape and without motion to the left or to the right.

From the graph the roots of $x^2 + 6x + 5 = 0$ are seen to be -5 and -1. These results are obtained from factoring; $x^2 + 6x + 5 = 0$, or $(x + 5)(x + 1) = 0$. Whence $x = -5$ or -1.

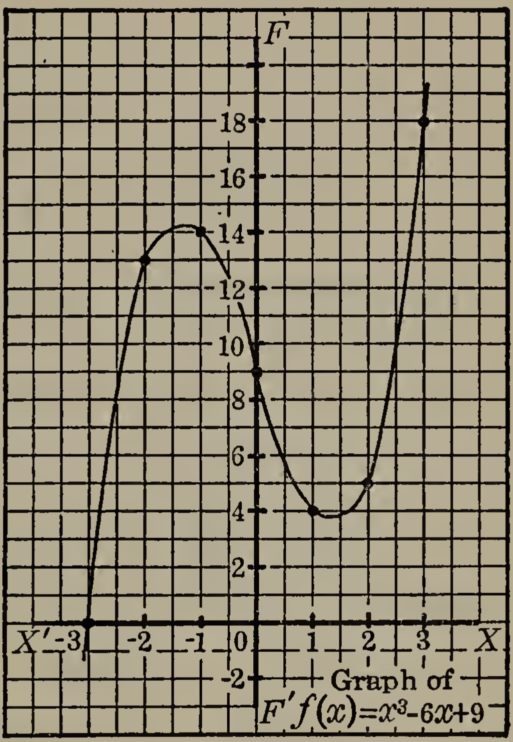
If we imagine curve (1) to move upward, the two roots change in value and become the single root of curve (2), which touches the x-axis at a point where x equals -3. Solving $x^2 + 6x + 9 = 0$ by factoring gives $(x + 3)(x + 3) = 0$. Whence $x = -3$.

If we now imagine curve (1) to move still farther upward from its position (2), it will no longer cut the x-axis. Therefore, when the curve reaches the position of (3), it does not cut the x-axis at all, and hence cannot be expected to show the values of the roots of the equation $x^2 + 6x + 13 = 0$, as, in fact, it does not. The graph does show, however, that the value of $x^2 + 6x + 13$ at the lowest point of the curve is 4. This means that for every real value of x, positive or negative, $x^2 + 6x + 13$ is never less than 4. The graph of (3) makes clear that no real number if substituted for x will make $x^2 + 6x + 13$ equal zero.

It can be shown by the method of section 196 that the roots of $x^2 + 6x + 13 = 0$ are $-3 + 2\sqrt{-1}$ and $-3 - 2\sqrt{-1}$.

NOTE. It required the genius of no less a man than Sir Isaac Newton first to observe from the graph of a function that two of its roots become imaginary simultaneously. He also saw that an equation with two of its roots equal to each other is, in a certain sense, the limiting case between equations in which the corresponding roots appear as two real and distinct roots and those in which they appear as imaginary roots.

213. Imaginary roots for a cubic equation. If we attempt to solve $x^3 - 6x + 9 = 0$ graphically, we obtain the graph of the accompanying figure. The curve crosses the x-axis at $x = -3$. This is the only real root the equation has; the other two are imaginary. The roots can here be obtained by factoring; thus $(x+3)(x^2-3x+3)=0$. The roots of $x^2-3x+3=0$, $\dfrac{3 \pm \sqrt{-3}}{2}$, are the two imaginary roots of the *cubic* equation.



EXERCISES

As far as possible solve graphically, finding roots to one decimal place:

1. $x^3 - x^2 - x - 2 = 0$. **2.** $x^3 + x^2 + x + 1 = 0$.

3. $x^3 - 7x - 6 = 0$.

4. $x^3 - 3x^2 + 5x - 15 = 0$.

5. $x^3 + 2x^2 - 11x + 20 = 0$.

6. $x^3 + 2x^2 - 1 = 0$.

7. $x^3 + 4x^2 - 8x + 24 = 0$.

8. $x^4 - 16 = 0$.

9. $x^4 - 9x^2 + x + 8 = 0$.

10. $x^4 - 8x^2 + 5 = 0$.

CHAPTER XXXI

THEORY OF QUADRATIC EQUATIONS

214. Formation of equations with given roots. According to section 80, the equation $(x-4)(x-1)=0$ has the roots 4 and 1. In general, the equation $(x-r_1)(x-r_2)=0$ has the roots r_1 and r_2, because either of these numbers, when substituted for x, satisfies the equation. Hence we can always find an equation whose roots are two given numbers r_1 and r_2 by setting the product of the binomials $x-r_1$ and $x-r_2$ equal to zero.

For example, an equation whose roots are 5 and 8 is seen in $(x-5)(x-8)=0$, or $x^2-13x+40=0$.

EXERCISES

Form equations whose roots are the following:

1. $3, 4$.

2. $1, 5$.

3. $2, -1$.

4. $-3, -5$.

5. $3, \frac{1}{2}$.

6. $7, \frac{3}{4}$.

7. $-\frac{2}{5}, \frac{3}{7}$.

8. $-\frac{3}{10}, -\frac{2}{7}$.

9. $3+\sqrt{2}, 3-\sqrt{2}$.

10. $2\pm\sqrt{5}$.

11. $1-\sqrt{-3}, 1+\sqrt{-3}$.

12. $-8+\sqrt{-5}, -8-\sqrt{-5}$.

13. $\frac{1}{3}+\sqrt{\frac{4}{3}}, \frac{1}{3}-\sqrt{\frac{4}{3}}$.

14. $\frac{1}{4}+\sqrt{-\frac{3}{2}}, \frac{1}{4}-\sqrt{-\frac{3}{2}}$.

15. 3, 6, 8.

HINT. $(x-3)(x-6)(x-8)=0$.

16. 2, 7, 3.

17. 2, -3, 8.

18. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

19. $\sqrt{-3}, -\sqrt{-3}, 3$.

20. $2+\sqrt{5}, 2-\sqrt{5}, 6$.

21. $2, -2, \sqrt{-1}, -\sqrt{-1}$.

22. $\sqrt{3}-\sqrt{5}, \sqrt{3}+\sqrt{5}, -\sqrt{3}-\sqrt{5}, -\sqrt{3}+\sqrt{5}$.

215. Relations between roots and coefficients of $x^2 + bx + c = 0$. By direct multiplication we obtain from

$$(x-r_1)(x-r_2)=0 \tag{1}$$

the equation $x^2-(r_1+r_2)x+r_1r_2=0. \qquad (2)$

Since r_1 and r_2 are the roots of (1), it appears from an inspection of (2) that the quadratic equation

$$x^2+bx+c=0$$

has the roots r_1 and r_2, provided $b=-(r_1+r_2)$ and $c=r_1r_2$.

For example, we may form at once the equation whose roots are 3 and 5, as follows:

$$x^2-(3+5)x+3\cdot 5=0, \text{ or } x^2-8x+15=0.$$

Similarly for the cubic equation $x^3+bx^2+cx+d=0$, roots are r_1, r_2, and r_3, we have $(x-r_1)(x-r_2)(x-r_3)=0$.

Then $\qquad b=-(r_1+r_2+r_3)$,

$\qquad c=r_1r_2+r_1r_3+r_2r_3$,

and $\qquad d=-r_1r_2r_3$.

c

EXERCISES

Form equations whose roots are the following:

1. $3, 8.$
2. $4, 9.$
3. $-2, 5.$
4. $-2, -3.$
5. $-8, -1.$
6. $2, 3, -4.$
7. $1, -2, 5.$
8. $\sqrt{2}, -\sqrt{2}.$
9. $3\sqrt{5}, -3\sqrt{5}.$
10. $2+\sqrt{3}, 2-\sqrt{3}.$
11. $1+\sqrt{7}, 1-\sqrt{7}.$

216. Relation between roots and coefficients of $ax^2+bx+c=0$. We will now show the relations which exist between the roots and the coefficients of the *general quadratic equation* $ax^2+bx+c=0$. By section 196 the roots of $ax^2+bx+c=0$ are

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Adding r_1 and r_2, we have

$$r_1+r_2=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}.$$

Therefore $$-(r_1+r_2)=\frac{b}{a}.$$

Multiplying r_1 by r_2, we have

$$r_1r_2=\frac{-b+\sqrt{b^2-4ac}}{2a}\cdot\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Therefore $$r_1r_2=\frac{c}{a}.$$

These results may be expressed verbally as follows:

For the equation $ax^2 + bx + c = 0$,

I. *The sum of the roots with its sign changed is* $\frac{b}{a}$.

II. *The product of the roots is* $\frac{c}{a}$.

These relations frequently afford the simplest means of checking the solution of a quadratic equation, as illustrated in Exercises 18–25, below.

EXERCISES

Form equations whose roots are the following:

1. $\frac{4}{3}, -\frac{3}{2}$.

Solution.

$$-\left(\frac{4}{3} - \frac{3}{2}\right) = \frac{1}{6} = \frac{b}{a}.$$

$$\left(\frac{4}{3}\right)\left(-\frac{3}{2}\right) = -2 = \frac{c}{a}.$$

Hence the required equation is

$$x^2 + \tfrac{1}{6}x - 2 = 0$$

or

$$6x^2 + x - 12 = 0.$$

2. $\frac{8}{3}, \frac{1}{4}$.

3. $.5, -.75$.

4. $1\frac{2}{3}, -4\frac{1}{2}$.

5. $4.75, 5.25$.

6. $3 + 2\sqrt{2}, 3 - 2\sqrt{2}$.

7. $9 + \sqrt{-3}, 9 - \sqrt{-3}$.

8. $\frac{4}{3} + \sqrt{6}, \frac{4}{3} - \sqrt{6}$.

9. $\frac{1}{4} + \frac{1}{4}\sqrt{-7}, \frac{1}{4} - \frac{1}{4}\sqrt{-7}$.

10. $1\frac{7}{10} \pm \sqrt{10}$.

11. $\pm\sqrt{-1} + 2$.

12. $-b, \frac{1}{b}$.

13. $\frac{2a}{3}, \frac{5a}{3}$.

14. $a + 1, a - 1$.

15. $\frac{m+n}{m-n}, \frac{m-n}{m+n}$.

16. $3, 7, \frac{1}{3}$.

17. $5, -5, \frac{1}{5}$.

Solve the following equations by the use of the formula and check by the use of I and II above:

18. $x^2 - 5x + 4 = 0.$
19. $x^2 - 2x - 5 = 0.$
20. $x^2 - x - 7 = 0.$
21. $x^2 - 5x - 10 = 0.$
22. $x^2 + 2x + 1 = 0.$
23. $x^2 + 4x + 4 = 0.$
24. $x^2 + 3x + 3 = 0.$
25. $3x^2 + 2x - 7 = 0.$

Find the value of the literal coefficient in the following:

26. $x^2 - 4x + a = 0$, if one root is 3.

HINT. The sum of the roots = 4.

27. $x^2 - 2x - c = 0$, if one root is 4.

28. $x^2 + cx - 15 = 0$, if one root is 5.

29. $x^2 + 2bx + 10 = 0$, if one root is -2.

30. $x^2 + 6x + a = 0$, if one root is twice the other.

31. $x^2 + 5x - c = 0$, if one root exceeds the other by 3.

32. $x^2 - 3x + b = 0$, if the difference between the roots is 7.

217. Character of the roots of a quadratic equation. It is often desirable to determine whether the roots of a given quadratic equation are **real** or **imaginary, rational** or **irrational, equal** or **unequal,** without solving the equation. This can be accomplished by use of the formulas for the roots of the quadratic $ax^2 + bx + c = 0$:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{1}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

These expressions are seen to differ from each other only in the sign preceding the radical. The expression

$$b^2 - 4ac,$$

which appears under the radical sign, is called the **discriminant** of the quadratic. The only way in which r_1 or r_2 can be a complex number is for the discriminant to be negative, since this is the only part of the root under a radical sign. If all of the coefficients a, b, and c are rational, r_1 or r_2 can be rational only when $\sqrt{b^2 - 4ac}$ is rational; that is, when the discriminant is a perfect square.

Hence if a, b, and c are rational, an inspection of (1) and (2) shows that the following statements are true:

I. *If $b^2 - 4ac$ is positive and not a perfect square, the roots are real, unequal, and irrational.*

For example, in $x^2 - 6x + 2 = 0$ the discriminant $b^2 - 4ac$ equals $(-6)^2 - 4 \cdot 1 \cdot 2 = 28$, which is not a perfect square. The roots of the equation are the real, unequal, and irrational numbers $3 \pm \sqrt{7}$.

II. *If $b^2 - 4ac$ is positive and a perfect square, the roots are real, unequal, and rational.*

For example, in the equation $3x^2 - 5x + 2 = 0$ the discriminant $b^2 - 4ac$ equals $(-5)^2 - 4 \cdot 3 \cdot 2 = 1$, which is a perfect square. The roots are the real, unequal, rational numbers $\frac{2}{3}$ and 1.

III. *If $b^2 - 4ac$ is zero, the roots are real and equal.*

For example, in the equation $9x^2 - 12x + 4 = 0$, the discriminant $b^2 - 4ac$ equals $(-12)^2 - 4 \cdot 9 \cdot 4 = 144 - 144 = 0$. Here the only number which satisfies this equation is $\frac{3}{2}$, so in one sense the equation has only one root. But since the left member has two identical factors each of which affords the same root of the equation, it is customary to say that the equation has two equal roots.

IV. *If* $b^2 - 4ac$ *is negative, the roots are imaginary.*

For example, in the equation $3x^2 - 7x + 5 = 0$ the discriminant $b^2 - 4ac$ equals $(-7)^2 - 4 \cdot 3 \cdot 5 = 49 - 60 = -11$. The roots of the equation are the conjugate imaginaries $\frac{7+\sqrt{-11}}{6}$ and $\frac{7-\sqrt{-11}}{6}$.

ORAL EXERCISES

Through the use of the discriminant determine the character of the roots of the following equations:

1. $x^2 + 2x + 1 = 0$.
2. $x^2 - 3x + 2 = 0$.
3. $2x^2 - 5x - 3 = 0$.
4. $4x^2 + 3x - 7 = 0$.
5. $7x^2 - 5x + 3 = 0$.
6. $2x^2 - 2x + 1 = 0$.
7. $x^2 - x + 1 = 0$.
8. $2x^2 - 5x + 7 = 0$.
9. $5x^2 + 5x - 1 = 0$.
10. $3x^2 - 4x + 5 = 0$.
11. $5x^2 + 6x - 12 = 0$.
12. $18x^2 + 36x + 25 = 0$.
13. $6x^2 + 11x + 3 = 0$.
14. $6x^2 + 9x - 60 = 0$.

EXERCISES

Determine the value of k which will make the roots of the following equations equal:

1. $x^2 + kx + 25 = 0$.

Solution. $a = 1$, $b = k$, $c = 25$.

Hence $b^2 - 4ac = k^2 - 100$.

For the roots to be equal $b^2 - 4ac$ must equal zero.

Therefore $k^2 - 100 = 0.$

Whence $k = \pm 10.$

Check. Substituting 10 for k in the original equation,

$$x^2 + 10x + 25 = (x + 5)^2 = 0.$$

Whence $x = -5$ only.

Similarly, substituting -10 for k,

we obtain $x^2 - 10x + 25 = (x - 5)^2 = 0.$

Whence $x = +5$ only.

2. $x^2 - kx + 4 = 0.$

3. $x^2 + 16x + k = 0.$

4. $4x^2 + 8x + k = 0.$

5. $x^2 - 4kx + 64 = 0.$

6. $2x^2 - kx + 2 = 0.$

7. $4x^2 + 12x + k - 1 = 0.$

8. $kx^2 + 42x + 49 = 0.$

9. $kx^2 - 36x + 36 = 0.$

10. $36x^2 - (k + 5)x + 25 = 0.$

11. $(k^2 + 13)x^2 + 112x + 64 = 0.$

CHAPTER XXXII

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

218. Graph of a quadratic equation in two variables. Before solving graphically a quadratic system, the method of graphing *one* quadratic equation in *two* variables must be clearly understood.

EXAMPLES

1. Construct the graph of $x^2 = 5y$.

Solution. Solving for y, we get

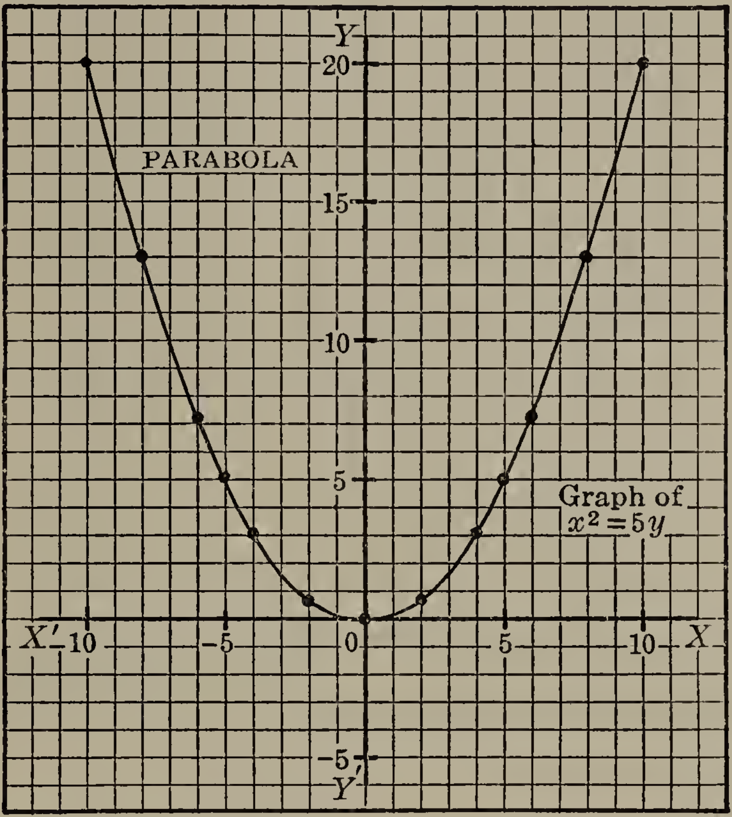
$$y = \frac{x^2}{5}.$$

We now assign values to x and then compute the approximate corresponding values of y. Tabulating the results gives:

If $x =$	10	8	6	5	4	3	2	1	0	-2	-4	-5	-6	-8	-10
Then $y =$	20	$12\frac{4}{5}$	$7\frac{1}{5}$	5	$3\frac{1}{5}$	$1\frac{4}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	0	$\frac{4}{5}$	$3\frac{1}{5}$	5	$7\frac{1}{5}$	$12\frac{4}{5}$	20

Using an x-axis and a y-axis as in graphing linear equations, plotting the points corresponding to the real numbers in the table, and drawing the curve determined by these points, we obtain the graph of the adjacent figure. The curve is called a **parabola**. The parabola, along with certain other graphs treated in this chapter, belongs to a family of curves known as **conic sections**. These will be discussed in more detail later.

The graph of any equation of the form $x^2 = ay$ is a parabola. This is a special case of the quadratic function considered in section 208.



2. Graph the equation $xy + 3 = 0$.

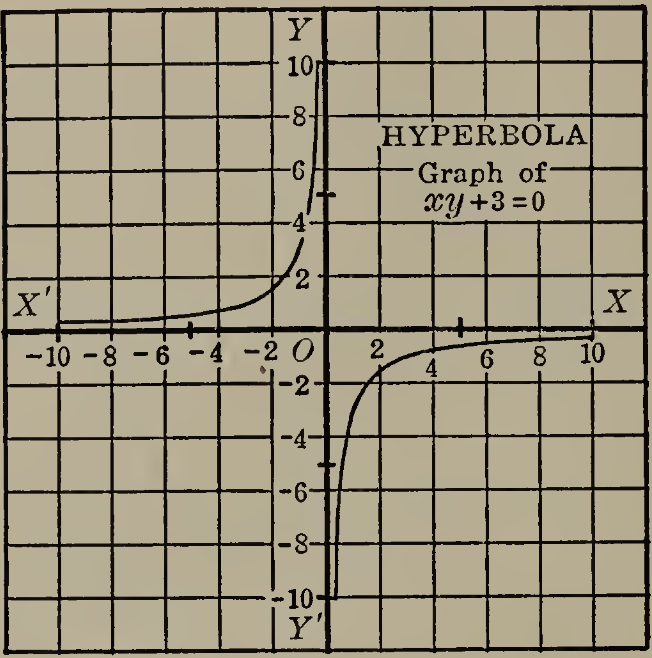
Solution. Solving for y, $y = -\frac{3}{x}$.

Assigning values to x as indicated in the following table, we then compute the corresponding values of y:

If $x=$	-10	-8	-6	-4	-3	-2	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	2	3	4	6	8	10
Then $y=$	$\frac{3}{10}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	3	4	6	12	-12	-6	-4	-3	$-1\frac{1}{2}$	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{3}{8}$	$-\frac{3}{10}$

Proceeding as before with the numbers in the table, we obtain the two-branched curve of the figure below, which does not touch either axis. The curve is called a **hyperbola**.

The graph of any equation of the form $xy = K$ is a *hyperbola.* The curve for $xy = K$ (K = any constant) is always in the same general position; that is, if K is positive, one branch of the curve lies in the first quadrant and the other branch in the third. If K is negative, one branch lies in the second quadrant and the other in the fourth.



3. Graph the equation $x^2 + y^2 = 25$.

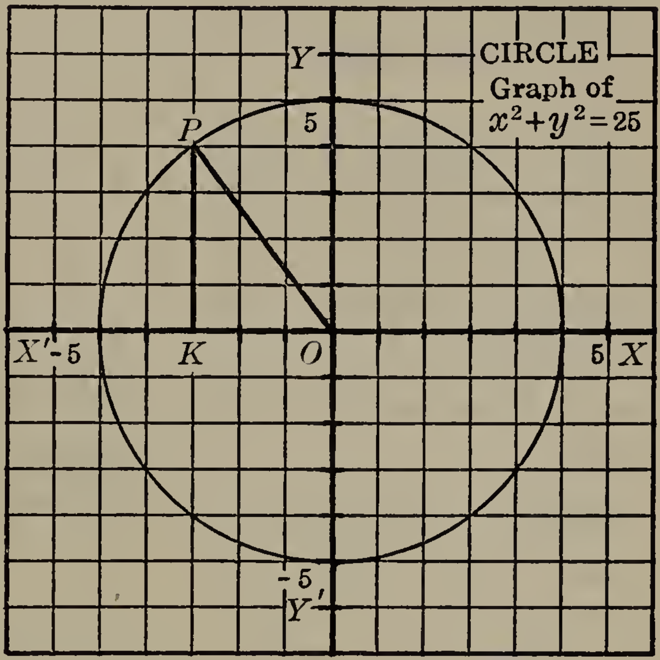
Solution. Solving for y, $y = \pm\sqrt{25 - x^2}$.

Assigning values to x as indicated in the following table, we obtain from page 330 the corresponding values of y:

If $x =$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
Then $y =$	$\pm\sqrt{-11}$	0	± 3	± 4	± 4.58	± 4.90	± 5	± 4.90	± 4.58	± 4	± 3	0	$\pm\sqrt{-11}$

For values of x numerically greater than 5 it appears that y is imaginary. The points corresponding to the pairs of real numbers in the table lie on the circle in the accompanying figure. The center of the circle is at the origin, and the radius is 5.

The graph of any equation of the form $x^2 + y^2 = r^2$ is a circle whose radius is r. This can be proved from the right triangle *PKO*. If *P* represents *any* point on the circle, *OK* equals the x-distance of *P*, *KP* equals the y-distance, and *OP* equals the radius. Now $\overline{OK}^2 + \overline{KP}^2 = \overline{OP}^2$; that is, $x^2 + y^2 = r^2$. It follows, then, that the graphs of $x^2 + y^2 = 16$ and $x^2 + y^2 = 10$ are circles whose centers are at the origin and whose radii are 4 and $\sqrt{10}$ respectively. Hereafter, when it is required to graph an equation of the form $x^2 + y^2 = r^2$, the student may use compasses and, with the origin as center and with the proper radius (the square root of the constant term), describe the circle at once.



In all the graphical work which follows, the student will save time by obtaining from the table on page 620 the square roots or cube roots which he may need.

4. Graph the equation $9x^2 + 4y^2 = 36$.

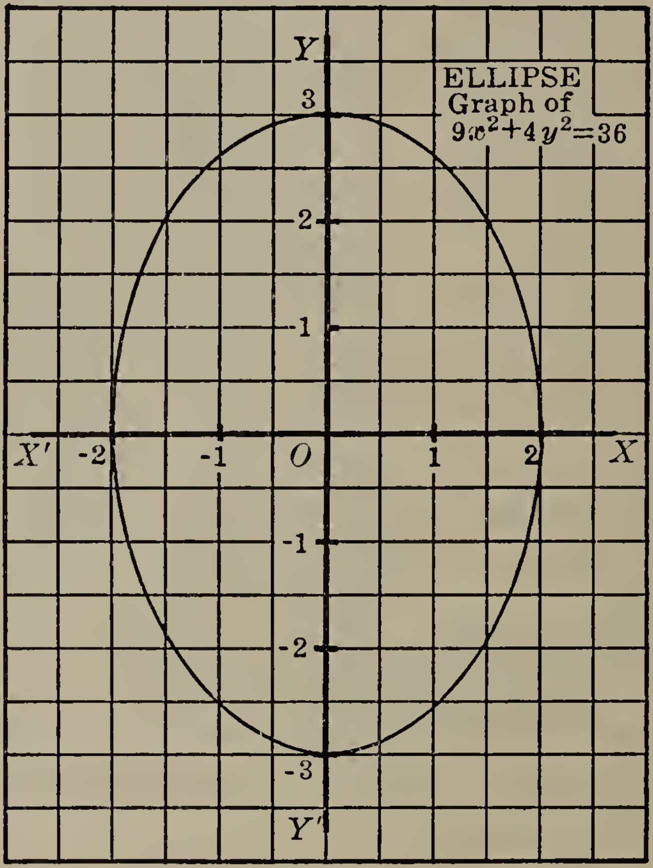
Solution. Solving for y,

$$y = \pm\sqrt{\frac{36 - 9x^2}{4}}$$
$$= \pm\tfrac{1}{2}\sqrt{36 - 9x^2}$$
$$= \pm\tfrac{3}{2}\sqrt{4 - x^2}.$$

Proceed as in Example 3:

If $x =$	-3	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3
Then $y =$	$\pm\frac{3}{2}\sqrt{-5}$	0	±1.98	±2.60	±2.90	±3	±2.90	±2.60	±1.98	0	$\pm\frac{3}{2}\sqrt{-5}$

For any value of x numerically greater than 2, y is imaginary. The points corresponding to the real numbers in the table lie on the graph of the adjacent figure. The curve is called an ellipse.



The graph of any equation of the form of $ax^2 + by^2 = c$ in which a and b are unequal and of the same sign as c is an *ellipse.*

NOTE. These three curves — the ellipse, the hyperbola, and the parabola — were first studied by the Greeks, who proved that they are the sections which one obtains by cutting a cone by a plane. Not for hundreds of years afterwards did anyone imagine that these curves actually appear in nature, for the Greeks regarded them merely as geometrical figures, and not at all as curves that have anything to do with our everyday life. One of the most important discoveries of astronomy was made by Kepler (1571–1630), who showed that the earth revolves around the sun in an ellipse, and stated the laws which govern the motion. Those comets that return to our field of vision periodically also have elliptic orbits, while those that appear once, never to be seen again, describe parabolic or hyperbolic paths.

The path of a ball thrown through the air in any direction, except vertically upward or downward, is a parabola. The approximate parabola which a projectile actually describes depends on the elevation of the gun (the angle with the horizontal), the quality of the powder, the amount of the charge, the direction of the wind, and various other conditions. This makes gunnery a complex subject.

EXERCISES

Construct the graphs of the following equations, and state the name of each curve obtained:

1. $x^2 = 6y$.	**6.** $xy = 10$.
2. $y^2 + 3x = 0$.	**7.** $xy = -8$.
3. $x^2 + y^2 = 4$.	**8.** $16x^2 + 25y^2 = 400$.
4. $x^2 + y^2 = 15$.	**9.** $9x^2 - 4y^2 = 36$.
5. $x^2 - y^2 = 36$.	**10.** $16x^2 + 9y^2 = 144$.

219. Graphical Solution of Quadratic Systems. That we may solve a system of two quadratic equations by a method similar to that employed in section 167 for linear equations appears from the

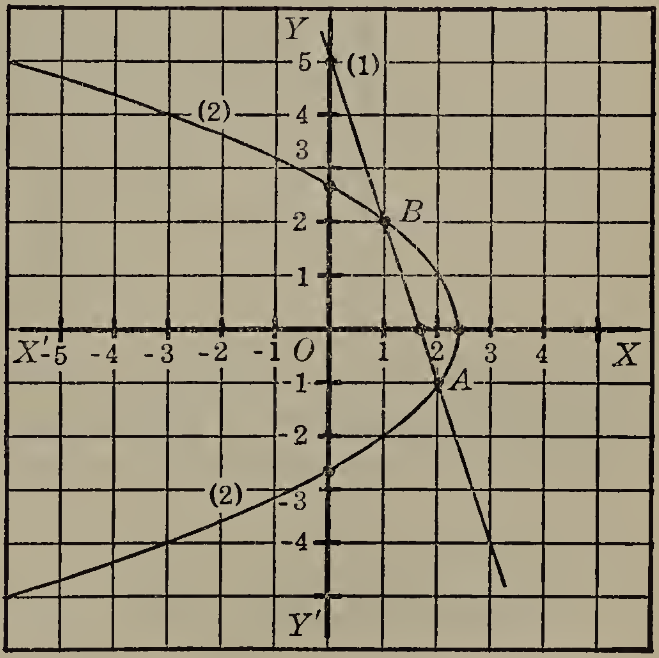
EXAMPLES

1. Solve graphically $\begin{cases} 3x + y = 5, & (1) \\ y^2 + 3x = 7. & (2) \end{cases}$

Solution. Constructing the graphs of (1) and (2), we obtain the straight line and the parabola shown in the adjacent figure. There are two sets of roots corresponding to two points of intersection, which are

$$A \begin{cases} x = 2, \\ y = -1 \end{cases}$$

and $B \begin{cases} x = 1, \\ y = 2. \end{cases}$

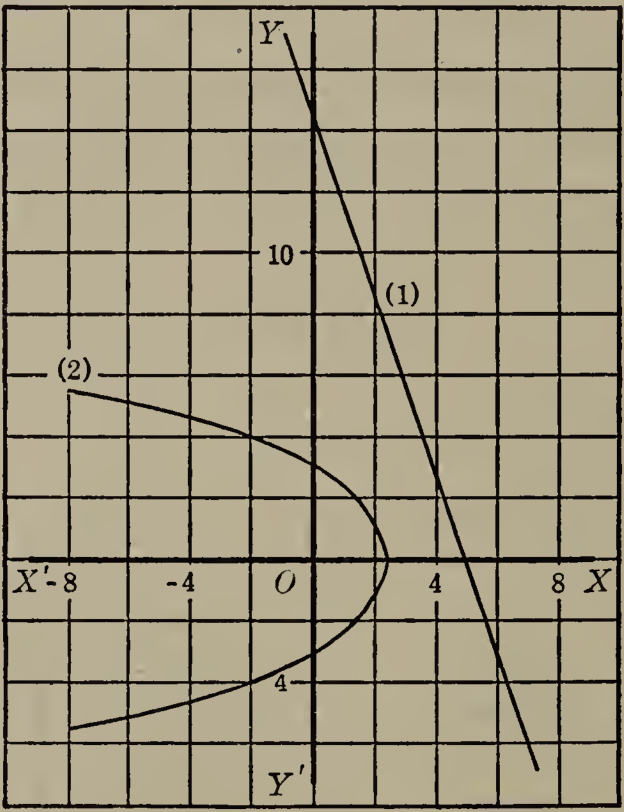


Note. If the straight line in the adjacent figure were moved to the right in such a way that it always remained parallel to its present position, the points A and B would approach each other and finally coincide. The line would then be tangent to the parabola.

Were the straight line moved still farther, it would neither touch nor intersect the parabola and there would be no graphical solution (see page 475).

2\. Solve graphically
$$\begin{cases} 3x + y = 15, & (1) \\ y^2 + 3x = 7. & (2) \end{cases}$$

Solution. The graphs of (1) and (2) are the straight line and the parabola of the adjacent figure. These curves have no real points of intersection. There are, however, two pairs of imaginary roots, for as will be seen in the next chapters solving (1) and (2) by substitution gives

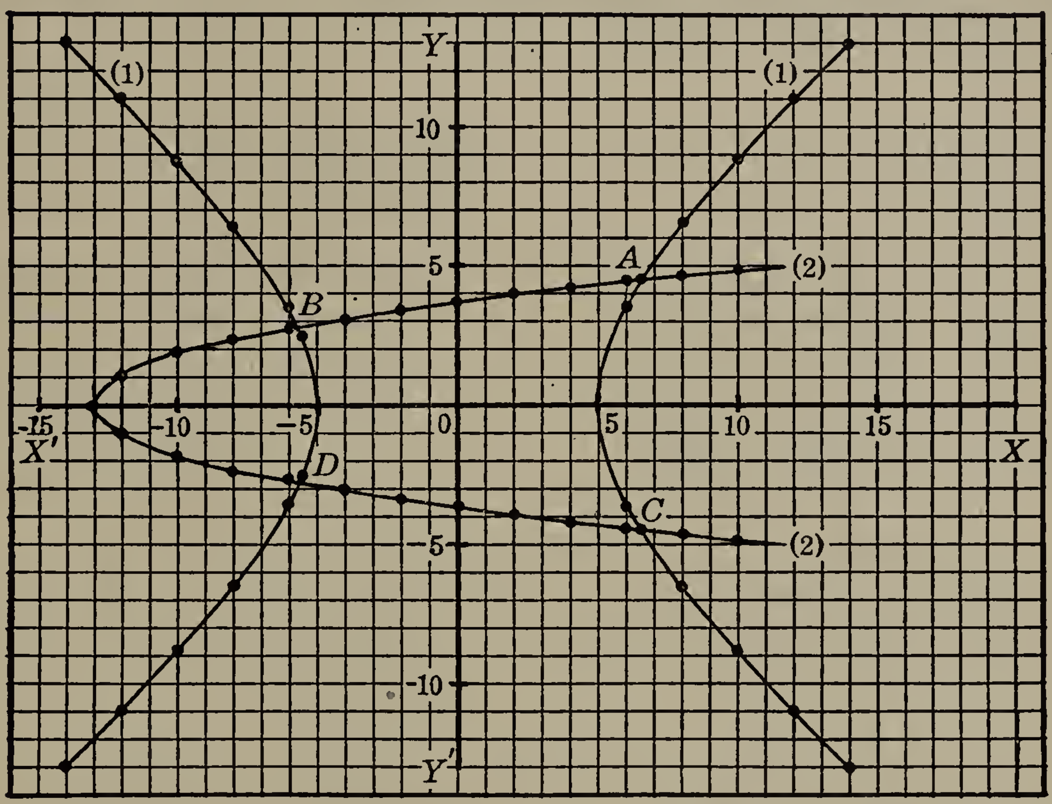


$$A\begin{cases} x = \frac{29}{6} - \frac{1}{6}\sqrt{-31}, \\ y = \frac{1}{2} + \frac{1}{2}\sqrt{-31}. \end{cases} \qquad B\begin{cases} x = \frac{29}{6} + \frac{1}{6}\sqrt{-31}, \\ y = \frac{1}{2} - \frac{1}{2}\sqrt{-31}. \end{cases}$$

The essential point to be emphasized here is that real roots of a simultaneous system correspond to real intersections, and imaginary roots correspond to no intersections of real graphs.

3. Solve graphically $\begin{cases} x^2 - y^2 = 25, & (1) \\ y^2 - x - 12 = 0. & (2) \end{cases}$

Solution. Constructing the graphs of (1) and (2), we obtain the hyperbola and the parabola of the adjacent figure. There are four sets of roots corresponding to the four points of intersection, which are approximately



$$A\begin{cases} x = 6.6, \\ y = 4.3. \end{cases} \qquad B\begin{cases} x = -5.6, \\ y = 2.5. \end{cases}$$

$$C\begin{cases} x = 6.6, \\ y = -4.3. \end{cases} \qquad D\begin{cases} x = -5.6, \\ y = -2.5. \end{cases}$$

If the two curves had been so chosen as to intersect only twice, their equations would have had only two sets of real roots.

Examples 1, 2, and 3 partially illustrate the truth of the following statement:

If in a system of two equations in two variables one equation is of the mth degree and one of the nth, there are *usually mn* sets of roots (real or imaginary) *and never more than mn such sets.*

EXERCISES

If possible solve graphically each of the following systems:

1. $\begin{cases} y^2 = 9x, \\ 2x + y = 6. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 4, \\ y + 2x = 7. \end{cases}$

3. $\begin{cases} xy = 8, \\ 2x + 3y = 10. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 9, \\ x^2 + y^2 = 4. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 16, \\ x + y = 12. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 25, \\ x^2 - y^2 = 16. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 36, \\ x^2 - y^2 = 144. \end{cases}$

8. $\begin{cases} xy = 9, \\ 2x + 3y = 4. \end{cases}$

9. $\begin{cases} x^2 = 9y, \\ 4x^2 + y^2 = 4. \end{cases}$

10. $\begin{cases} x^2 + 3y = 15, \\ x + 3y = 10. \end{cases}$

11. $\begin{cases} x^2 + 3y = 15, \\ x + 3y = 25. \end{cases}$

12. $\begin{cases} x^2 + y = 7, \\ y^2 + x = 4. \end{cases}$

13. $\begin{cases} x - y\sqrt{3} = 0, \\ x^2 = y^3 - 16y. \end{cases}$

220. Graphical presentation of numerical data. A great variety of statistics can be presented graphically in a very striking manner. Business and commercial houses have during the past few years used the method extensively not only to present facts but also to aid in interpreting them and in indicating their meaning.

The following exercises illustrate one of the most striking and important features of the graphical presentation of numerical data.

EXERCISES

1. For the year 1918 personal incomes between $4000 and $15,000 in the United States were grouped as follows:

Annual Income	Number of Persons (thousands)	Annual Income	Number of Persons (thousands)
$4,000 to $5,000	430	$10,000 to $11,000	36
5,000 to 6,000	235	11,000 to 12,000	28
6,000 to 7,000	143	12,000 to 13,000	22
7,000 to 8,000	95	13,000 to 14,000	18
8,000 to 9,000	67	14,000 to 15,000	15
9,000 to 10,000	48		

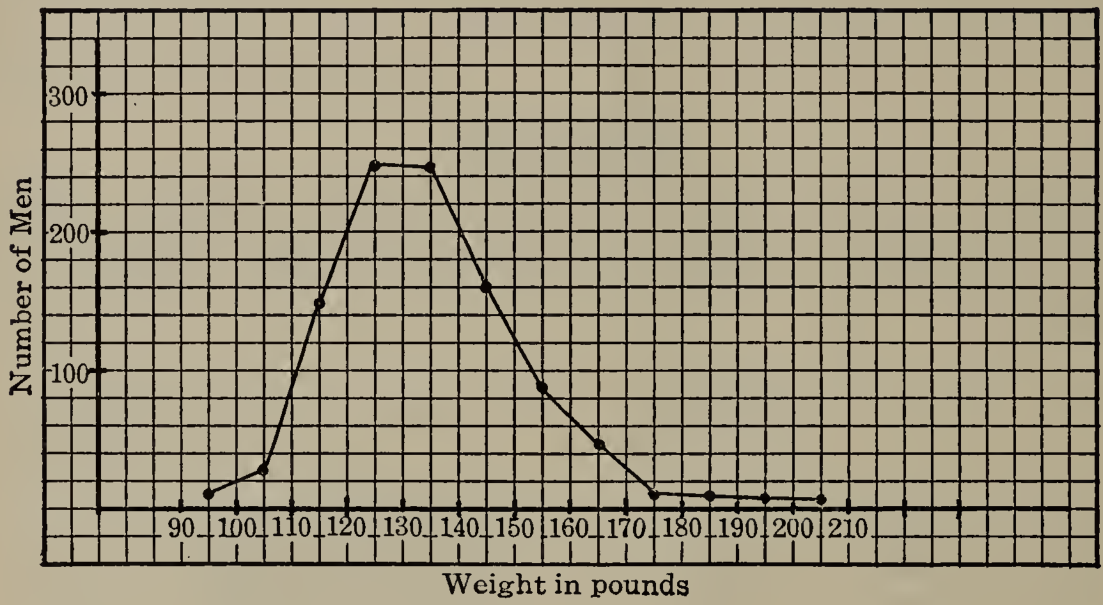
Graph these figures. From this curve estimate the number of persons having an income of $15,000 to $16,000 per year.

Hint. Plot the income on the horizontal axis, and the number of persons on the vertical axis.

2. The weights of 1000 men chosen at random were distributed as follows:

Weight (Lbs.)	Number of Men	Weight (Lbs.)	Number of Men
90 to 100	13	150 to 160	89
100 to 110	28	160 to 170	46
110 to 120	146	170 to 180	18
120 to 130	245	180 to 190	9
130 to 140	242	190 to 200	3
140 to 150	160	200 to 210	1

Plot these figures, using the vertical axis for the number of men in each weight group.



Curves of this general form are very characteristic of certain classes of physical measurement. A few instances are given in the following exercises.

3. Measurements were taken on the lengths of pods of a certain variety of bean. The following figures were obtained from this study.

Length of Pod (quarter inches)	Number	Length of Pod (quarter inches)	Number
12	2	18	85
13	7	19	81
14	25	20	56
15	40	21	13
16	63	22	4
17	92	23	1

Draw the length distribution curve for this set of figures and compare it with the preceding curve.

4. In a certain college the class marks in English courses were distributed in the following manner:

Class Mark	Number	Class Mark	Number
20	21	60	45
25	21	65	27
30	42	70	28
35	54	75	11
40	68	80	8
45	64	85	3
50	78	90	3
55	27		

Plot these figures, and state whether they follow the same rule as the figures of the preceding exercises.

5. The following figures show the distribution of the heights of Japanese soldiers.

Height (inches)	Number of Soldiers	Height (inches)	Number of Soldiers
56	47	63	1,698
57	125	64	1,328
58	316	65	839
59	640	66	442
60	1,065	67	208
61	1,486	68	64
62	1,730	69	12

Plot these figures as indicated in Exercise 2. Is this curve of the same type as that for Exercise 2? Is it reasonable that height distribution should follow the same general law as does weight distribution?

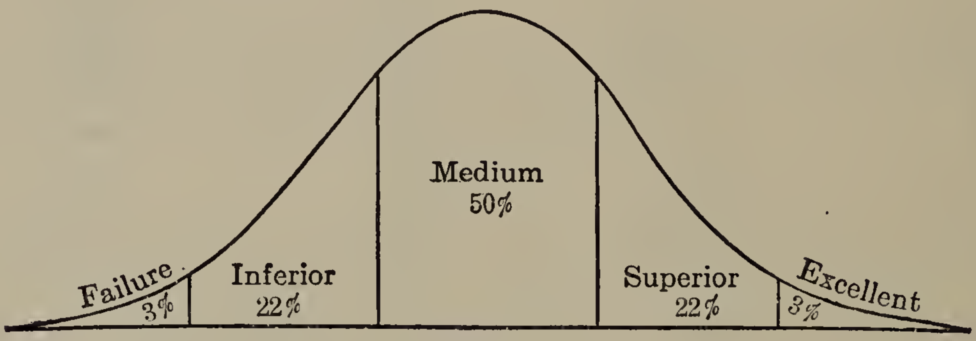
It may appear accidental that the foregoing measurements should group themselves with any regularity. But if the number of measurements of this type is large and each is made with care, they obey a law called the law of probability. In fact the graphs of the data in Exercises 4 and 5 are close approximations to what is called the probability curve.

The equation of this curve is $y = e^{-x^2}$ when $e = 2.7$ approximately.

6. Construct the graph of the equation $y = e^{-x^2}$ between the values $x = -2$ and $x = 2$.

HINT. Let $x = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0$, etc.

NOTE. It is well established that physical characteristics, such as those illustrated by the graphs of Exercises 2 and 3, obey the law of probability. The graph of Exercise 4 raises the question, Do mental characteristics also obey this law? An interesting aspect of this is given by the fact that an increasing number of high schools and colleges assume that such is the case, and grade, or mark, their students by a system based on the law of probability. Such a system assumes that if one hundred or more students in any subject are examined, the number of students and their degree of mastery of the subject arrange themselves according to the adjacent probability curve. Obviously between very many students the differences in grades attained will be small, between many others they will be moderate, and between only a few will they be great.



In the statistical study of problems which have their origin quite remote from each other, this curve frequently occurs, and occupies a central position in the mathematical theory of statistics.

CHAPTER XXXIII

SYSTEMS SOLVABLE BY QUADRATICS

221. Introduction. The algebraic solution of the system $x^2 + y = 7$ and $y^2 + x = 11$ leads to an equation of the fourth degree. For substituting in the second equation the value of y from the first gives $(7 - x^2)^2 + x = 11$ or $x^4 - 14x^2 + x + 38 = 0$. Similarly, given a pair of equations each of which is a general equation of the second degree such as $ax^2 + by^2 + cxy + dx + ey + f = 0$, we may regard one as a quadratic equation and solve it for x in terms of y and substitute the result in the other. This also gives an equation of the fourth degree which must then be solved for y, and after that is done the corresponding values x can be obtained. The algebraic solution of such systems will not be possible for the student without an advanced course in algebra. It should be noted, however, that by the methods of Chapter XXXII one can solve such systems as the above for real roots provided the terms have numeric coefficients. Moreover, many systems which frequently arise in practice can be solved algebraically since the result of substituting from one into the other gives an equation which can be solved by quadratics. Such systems will now be studied.

222. Linear and quadratic systems. Every system of equations in two variables in which one is of the first degree (linear) and the other is of the second degree (quadratic) can be solved by the method of substitution.

EXAMPLE

Solve the system
$$\begin{cases} 2x^2 - 3xy = 5, & (1)\\ 2x + 3y = 19. & (2)\end{cases}$$

Solution. Solving (2),
$$y = \frac{19 - 2x}{3}. \tag{3}$$

Substituting $\frac{19 - 2x}{3}$ for y in **(1),**
$$2x^2 - 3x \cdot \frac{(19 - 2x)}{3} = 5. \tag{4}$$

From (4),
$$4x^2 - 19x = 5. \tag{5}$$

Solving (5),
$$x = 5, -\tfrac{1}{4}. \tag{6}$$

Substituting 5 for x in (3),
$$y = \frac{19 - 10}{3} = 3. \tag{7}$$

Substituting $-\frac{1}{4}$ for x in (3),
$$y = \frac{19 + \frac{1}{2}}{3} = 6\tfrac{1}{2}. \tag{8}$$

Hence, for $x = 5, y = 3,$

and for
$$x = -\tfrac{1}{4}, y = 6\tfrac{1}{2}. \tag{9}$$

Check. Substituting 5 for x and 3 for y in (1) and (2):
$$50 - 45 = 5, \tag{10}$$
$$10 + 9 = 19. \tag{11}$$

Substituting $x = -\frac{1}{4}$ and $y = 6\frac{1}{2}$ in (1) and (2),
$$\tfrac{1}{8} + \tfrac{39}{8} = 5, \tag{12}$$
$$-\tfrac{1}{2} + \tfrac{39}{2} = 19. \tag{13}$$

The student should note that many pairs of values of x and y can be found which will satisfy either of the two original equations. Hence in checking it is necessary to substitute each set of roots in *both* equations in order to prove that the roots obtained are correct.

It will be found that many systems in two unknowns in which one is of the first degree (linear) and the other is of the third or even a higher degree can be solved by the method of substitution. Thus, for example, the system $x^3 - y^3 = 7$ and $x - y = 3$ can be solved by substitution. So also can the system $x^4 + y^4 = 17$ and $x + y = 3$. Obviously if the result of substitution gives an equation of the third or fourth degree which can be solved by factoring, the algebraic solution can be completely carried out.

Occasionally when neither equation is linear one can be solved for x in terms of y and the solution completed by substitution. In the system $\left\{\begin{array}{l} 2x^2 + 5xy + 3y^2 = 0 \\ x^2 + xy = 16 \end{array}\right\}$ the first equation can be solved for x in terms of y and then substitution in the second can be used to finish the solution.

EXERCISES

Solve the following systems, pair results, and check each set of roots.

1. $x^2 + y^2 = 68,$
$x = 4y.$

2. $x^2 + 5y = 29,$
$x + y = 7.$

3. $x + y = 9,$
$x^2 + y^2 = 41.$

4. $s^2 - t^2 = 5,$
$3s + t = 11.$

5. $x^2 + xy = 10,$
$x + 2y = 8.$

6. $y^2 - 3yz = 0,$
$2y + z = 7.$

7. $2y + 3yz = -2,$
$y - 3z = 5.$

8. $2y^2 + 3yz = 8,$
$3y + 2z = 7.$

9. $4s^2 - 3t^2 = 24,$
$2s - 3t = 12.$

10. $3s^2 - 4st = 80,$
$3s + 2t = 8.$

11. $s^2 + 5t^2 = 24,$
$4s - 3t = 14.$

12. $s^2 + st + t^2 = 9,$
$s + t = 0.$

13. $5y + z^2 = 3yz,$
$y - 2z = 0.$

14. $s^2 - st + t^2 = 84,$
$s + t = 12.$

15. $yz = 60,$
$4y + 3z = 56.$

16. $2y + 5z = 5,$
$yz + 4z = -42.$

17. $x_1 - 10x_2 = 22,$
$x_1x_2 - 6x_2 = -6.$

18. $\frac{s}{t} + \frac{t}{s} = \frac{5}{2}.$
$s + 2t = 8.$

19. $\frac{1}{x} - \frac{1}{z} = 4.$
$3x - 5z = 4.$

20. $\frac{x}{3} - \frac{y}{6} = 8,$
$\frac{9}{x} - \frac{4}{y} = \frac{5}{6}.$

21. $\frac{27}{x} + \frac{y}{3} = 1,$
$\frac{3}{y} - \frac{x}{18} = -1.$

22. $.8x + 3y = \frac{3}{2},$
$\frac{y}{x} = \frac{7}{5}.$

23. $\frac{s+2}{t+4} = \frac{1}{2},$
$\frac{t}{s-1} = \frac{s}{t-8}.$

24. $x + 10xy = 19.5,$
$x + 100y = 18.$

25. $\frac{t+1}{s-4} = \frac{s+1}{s-3},$
$\frac{10}{s} + \frac{4}{t} = 4.$

26. $\frac{162}{y} + \frac{21}{x} = 100,$
$2x - 6y = 5.$

27. $s^2 + t^2 - 2ts = 100,$
$s - 2t = 13.$

28. $x^3 - y^3 = 56,$
$x - y = 2.$

29. $v^2 + u^2 + 4u - 6v = 23,$
$2v - 5u = -14.$

30. $\frac{x^2}{9} + \frac{y^2}{16} = 2,$
$5x + 3y = 3.$

31. $yz + 2z^2 = 2a^2 - 2ac,$
$y - 2z = 2a + 2c.$

32. $xy = a^2,$
$x - y = a.$

33. $\frac{y}{a-z} + \frac{a-z}{y} = \frac{5}{2},$
$y + z = c.$

34. $yz = a(y - z),$
$a(y - z) = c(y + z).$

223. Homogeneous equations. If both the equations of a system are quadratic, an attempt to solve it by substitution gives an equation of the fourth degree. In most cases such an equation cannot be solved by factoring and its algebraic solution by any other method is beyond the student at present. With certain types of systems, however, which occur more or less frequently, we can employ special devices and avoid the solution of an equation of higher degree than a quadratic. Among these systems are the so-called "homogeneous" systems.

An equation is *homogeneous* if, on being written so that one member is zero, the terms in the other member are of the same *degree* with respect to the two variables.

Thus $5xy - x^2 = y^2$ and $x^2 + 8xy + 6y^2 = 0$ are homogeneous equations of the second degree. A homogeneous equation of the third degree is $3x^3 - x^2y + 4xy^2 = 8y^3$.

The system $\left\{\begin{matrix} 5xy - x^2 = y^2 \\ 2y^2 - 5xy - 2y^2 = 0 \end{matrix}\right\}$ is a homogeneous system.

Systems like $\left\{\begin{matrix} x^2 - xy + y^2 = 7 \\ 2x^2 + 3y^2 - xy = 4 \end{matrix}\right\}$ are often called homogeneous. As will be seen, the method of solving such systems is about the same as the method of solving a homogeneous system. Hence they are classed with homogeneous systems.

224. Systems having both equations quadratic. Occasionally when both equations are quadratic, the terms which occur in the two are so related that the elimination of terms involving both variables or of the constant terms can be performed. The system in the first example below is of such a type.

EXAMPLE

1. Solve the system $\begin{cases} 4x^2 - 3xy = 18, & (1) \\ x^2 + xy = 15. & (2) \end{cases}$

Solution. First eliminate xy by addition (section 103).

(2)· 3, $3x^2 + 3xy = 45$ (3)

(1)+(3), $7x^2 = 63.$ (4)

From (4) $x = \pm 3.$

Substituting 3 for x in (2), $9 + 3y = 15$, whence $y = 2.$

Substituting -3 for x in (2), $9 - 3y = 15$, whence $y = -2.$

Therefore $x = 3, -3,$

and $y = 2, -2.$

These values can be checked as usual.

The method of solving a system in which every term is of the second degree except the constant term is as follows:

2. Solve the system $\begin{cases} x^2 - xy + y^2 = 7, & (1) \\ y^2 - 2xy = -3. & (2) \end{cases}$

Solution. First we eliminate the constant terms, 7 and -3, in order to obtain a homogeneous equation.

(1) · 3, $3x^2 - 3xy + 3y^2 = 21.$ (3)

(2) · 7, $7y^2 - 14xy = -21.$ (4)

(3)+(4), $3x^2 - 17xy + 10y^2 = 0.$ (5)

Solving (5) for x in terms of y, $x = 5y.$ (6)

$$x = \frac{2y}{3}. \qquad (7)$$

Substituting $x = 5y$ in (2),

$$y^2 - 10y^2 = -3. \qquad (8)$$

Solving (8), $y = \pm\sqrt{\tfrac{1}{3}} = \pm\tfrac{1}{3}\sqrt{3}.$ (9)

Substituting from (9) in (6), $x = \pm \frac{5}{3}\sqrt{3}$. (10)

Substituting $\frac{2y}{3}$ for x in (2),

$$y^2 - \frac{4y^2}{3} = -3. \qquad (11)$$

Solving (11), $y = \pm 3$. (12)

Substituting from (12) in (7), $x = \pm 2$. (13)

When $x =$	2	-2	$\frac{5}{3}\sqrt{3}$	$-\frac{5}{3}\sqrt{3}$
Then $y =$	3	-3	$\frac{1}{3}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$

EXERCISES

Solve, pair results, and check:

1. $x^2 - y^2 = 5,$
$2x^2 + y^2 = 22.$

2. $3x^2 - 2xy = 8,$
$2x^2 + 3xy = 14.$

3. $x^2 - xy + y^2 = 7,$
$x^2 + xy - y^2 = 1.$

4. $3xy - x^2 = 8,$
$xy + 3y^2 = 20.$

5. $2t^2 + st = 5,$
$st + s^2 = 12.$

6. $2s^2 + t^2 = 41,$
$s^2 - 4t^2 = -20.$

7. $2s^2 + 3t^2 + 2 = 5,$
$5t^2 - 3s^2 - 2 = 3.$

8. $x^2 + 2y^2 = 33,$
$2x^2 - 3xy + y^2 = 24.$

9. $x^2 - 3xy + 2y^2 = 0,$
$3x^2 - 2xy + y^2 = 18.$

HINT. Solve the first equation for x in terms of y and substitute in the second.

10. $xy - y^2 = 0,$
$x^2 - 2xy = 1.$

11. $x^2 + y^2 = 125,$
$3x^2 + 2y^2 = 7xy.$

12. $x^2 + xy = 77,$
$xy + y^2 = 44.$

HINT. Add the two equations and then obtain $x + y = \pm 11$, etc.

13. $x^2 - xy = 24,$
$3xy - 4y^2 = -40.$

HINT. Subtract the second equation from the first, etc.

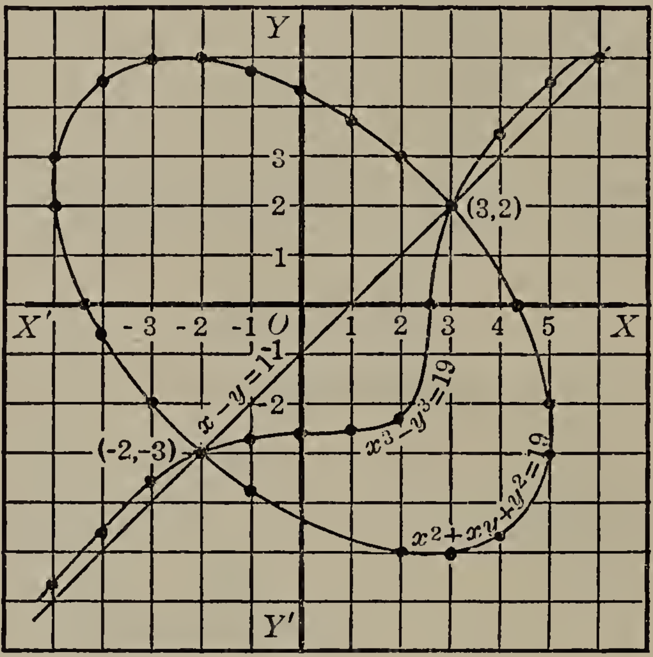
14. $x^2 + xy + y^2 = 52,$
$x^2 - xy + y^2 = 84.$

15. $s^2 + st = 2a^2,$
$t^2 + st = 2a^2.$

16. $s^2 + st = c,$
$t^2 + st = n.$

17. $xy - y^2 = 2ac,$
$x^2 - xy = a^2 + c^2.$

225. Equivalent systems. Equivalent systems of equations are systems which have the same set or sets of roots.



If the three systems

$$A.\ \begin{cases} x^3 - y^3 = 19, & (1) \\ x - y = 1. & (2) \end{cases} \qquad B.\ \begin{cases} x^2 + xy + y^2 = 19, & (3) \\ x - y = 1. & (2) \end{cases}$$

and

$$C.\ \begin{cases} x^3 - y^3 = 19, & (1) \\ x^2 + xy + y^2 = 19. & (3) \end{cases}$$

are solved, the values $\begin{Bmatrix} x = 3, & -2 \\ y = 2, & -3 \end{Bmatrix}$ are obtained for each.

Hence systems A, B, and C are equivalent.

The adjacent graphs of the three equations in the systems A, B, and C illustrate the equivalence of the three systems as well as the striking difference between them. In the graph the two common points of intersection are (3, 2) and (− 2, − 3). This shows that systems *A*, *B*, and *C* have the same sets of roots and hence are equivalent.

In this connection the student should remember that the equivalence of several systems, such as *A*, *B*, and *C*, depends solely on the fact that these systems have the same sets of roots. The form of the individual curves which go to make up the different systems does not in any degree determine whether the systems are equivalent. From this it follows that an unlimited number of equivalent systems can be set up with the roots, (3, 2) and (−2, −3), just as an unlimited number of curves can be drawn through the points (3, 2) and (−2, −3).

226. Symmetric systems. A system of equations in x and y is symmetric if the system is not altered by substituting x for y and y for x.

Thus $x + y = 5$ and $x^2 + xy + y^2 = 20$ form a symmetric system, but $x + y = 5$ and $x^2 + xy - y^2 = 20$ do not.

Certain symmetric systems or systems which are nearly so can be easily solved by the method of substitution. This is true of the following types:

$$\begin{cases} xy = 12, \\ x \pm y = 3. \end{cases} \qquad \begin{cases} x \pm 3\,y = 7, \\ x^2 + 9\,y^2 = 37. \end{cases} \qquad \begin{cases} x \pm y = 5, \\ x^2 + y^2 = 25. \end{cases}$$

A few other systems which are symmetric or nearly so are more easily solved by certain special methods. The following list contains typical systems, and the methods applicable are outlined in Exercises 1, 10, 12, which follow.

It should be noted that many systems can be solved by substitution even though neither equation of the system is linear. This is possible with the system $xy = 6$ and

$x^2 + y^2 = 25$. The solution of this system and of many somewhat similar systems can be more easily carried out, however, by the methods which follow.

EXERCISES

1. Solve $\begin{cases} x^2 + y^2 = 45, & (1) \\ xy = 18. & (2) \end{cases}$

HINT. Here the second equation could be solved for x in terms of y and the solution completed by substitution. A simpler method is to combine them in such a way as to obtain numeric values for $x + y$ and $x - y$ as follows:

$$\begin{array}{lrr} (2)\cdot 2, & 2\,xy = 36. & (3) \\ (1)+(3), & x^2 + 2\,xy + y^2 = 81. & (4) \\ \text{From } (4), & x + y = \pm 9. & (5) \\ (1)-(3), & x^2 - 2\,xy + y^2 = 9. & (6) \\ \text{From } (6), & x - y = \pm 3. & (7) \end{array}$$

(5) and (7) combined give the four linear systems:

$A.\ \begin{cases} x + y = 9, & (8) \\ x - y = 3. & (9) \end{cases}$ $\qquad C.\ \begin{cases} x + y = -9, & (12) \\ x - y = 3. & (13) \end{cases}$

$B.\ \begin{cases} x + y = 9, & (10) \\ x - y = -3. & (11) \end{cases}$ $\qquad D.\ \begin{cases} x + y = -9, & (14) \\ x - y = -3. & (15) \end{cases}$

These four systems are equivalent to the original one.

The solution of these systems is left to the student.

The pairs of roots found for the four systems A, B, C, D will check in the original system.

2. $x^2 + y^2 = 65$,
   $xy = 28$.

3. $4\,x^2 + y^2 = 104$,
   $xy = 10$.

4. $x^2 + xy + y^2 = 19$,
   $xy = 6$.

5. $x^2 + 9\,y^2 = 244$,
   $xy = 40$.

6. $x^2 - xy + y^2 = 21$,
   $xy = 20$.

7. $x^2 - xy + y^2 = 84$,
   $x^2 + xy + y^2 = 244$.

8. $x^2 + xy + 4y^2 = 64,$
$xy = 12.$

9. $4x^2 + 9y^2 = 72,$
$xy = 6.$

10. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 5,$
$\dfrac{1}{xy} = 2.$

Hint. To clear of fractions would merely increase the difficulty of solution. We can combine the equations more effectively as follows:

$$\frac{2}{xy} = 4.$$

Then $$\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 9.$$

Or $$\frac{1}{x} + \frac{1}{y} = \pm 3.$$

Similarly $$\frac{1}{x} - \frac{1}{y} = \pm 1.$$

Here as in Exercise 1 we have four systems equivalent to the original one. These four may be solved for $\frac{1}{x}$ and $\frac{1}{y}$ and thus finally the four sets of values of x and y obtained.

11. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 13,$
$\dfrac{1}{xy} = 6.$

12. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 25,$
$\dfrac{1}{x} - \dfrac{1}{y} = 1.$

Hint. $\dfrac{1}{x^2} - \dfrac{2}{xy} + \dfrac{1}{y^2} = 1.$

Then $\dfrac{2}{xy} = 24$, etc.

13. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 41,$
$\dfrac{1}{x} + \dfrac{1}{y} = 9.$

14. $\dfrac{1}{9x^2} + \dfrac{1}{4y^2} = 20,$
$\dfrac{1}{3x} + \dfrac{1}{2y} = -2.$

15. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6},$
$\dfrac{1}{xy} + \dfrac{1}{18} = 0.$

16. $\dfrac{1}{x} + \dfrac{1}{y} = a + b,$
$\dfrac{1}{xy} = ab.$

17. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 5a^2,$
$xy + \dfrac{1}{2a^2} = 0.$

227. Use of division in systems of equations. Sometimes an equation simpler than one of a given system can be derived by dividing the left and right members of one equation by the corresponding members of the other. The system composed of this derived equation and the simpler of the two given ones is equivalent to the original system and may be more easily solved.

EXERCISES

Solve, using division where possible, pair results, and check each set of real roots:

1. $$4x^2 - y^2 = 16,$$ $$2x - y = 8.$$

HINT. Division gives the equivalent system.

$$2x + y = 2,$$
$$2x - y = 8, \text{ etc.}$$

2. $x^2 - y^2 = 16,$
$x - y = 8.$

3. $25x^2 - 4y^2 = 0,$
$5x + 2y = 2.$

4. $x^3 + y^3 = 488,$
$x + y = 2.$

5. $x^3 - y^3 = 468,$
$x - y = 12.$

6. $u^3 + v^3 = 296,$
$u^2 - uv + v^2 = 148.$

7. $xy + y^2 = 12,$
$x^2 + xy = 24.$

HINT. $$\frac{y(x+y)}{x(x+y)} = \frac{12}{24},$$
or $$2y = x.$$

8. $8x^3 + y^3 = 16,$
$2x + y = 4.$

9. $s^3 - 8t^3 = 35,$
$s - 2t = 5.$

10. $\frac{1}{x} + \frac{1}{y} = 30,$
$\frac{1}{y^2} - \frac{1}{x^2} = 300.$

11. $s^2 - st = 120,$
$t^2 - st = 24.$

12. $\frac{1}{u^3} - \frac{1}{v^3} = 37,$
$\frac{1}{u} - \frac{1}{v} = 1.$

13. $x^4 + x^2y^2 + y^4 = 21,$
$x^2 + xy + y^2 = 7.$

14. $x^2 - y^2 = ab,$
$x + y = a.$

15. $x^3 + y^3 = a^3 - 8,$
$x + y = a - 2.$

MISCELLANEOUS EXERCISES

Solve by any method and pair results. If any system cannot be solved by methods previously given, solve it graphically.

1. $x + y = 4,$
$xy = 3.$

2. $x^2 - xy = -15,$
$2x - y = 14.$

3. $x^2 + xy + y^2 = 7,$
$2x + 3y = 0.$

4. $x^2 + y = 38,$
$x + 5y = 4.$

5. $s^2 - t^2 = 24,$
$s + 2t = 3.$

6. $x^2 + y^2 = 100,$
$xy = 48.$

7. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5,$
$x - y = 5.$

8. $u^3 + v^3 = 35,$
$u + v = 5.$

9. $\sqrt[3]{x} - \sqrt[3]{y} = 1,$
$x - y = 19.$

10. $yz + z = 20,$
$yz + y = 18.$

11. $x^3 - y^3 = 152,$
$x^2 + xy + y^2 = 19.$

12. $h^2 - hk = 160,$
$hk - k^2 = 96.$

13. $y - xy = 72,$
$3x + xy = -50.$

14. $s^2 + t^2 = 100,$
$s^2 - t^2 = 28.$

15. $x_1^2 + x_1x_2 = 200,$
$x_1x_2 + x_2^2 = -100.$

16. $x + y = 1.7,$
$xy = .52.$

17. $.1s + .1t = .015,$
$.2s^2 + .2t^2 = .0025.$

18. $x + .2y = .94,$
$.2x^2 - xy = -.742.$

19. $x + y = 2a,$
$xy = a^2 - b^2.$

20. $x^2 + xy + y^2 = 3a^2,$
$x^2 + xy = 2a^2.$

21. $m^{\frac{1}{2}} + n^{\frac{1}{2}} = 13,$
$m^{\frac{1}{2}}n^{\frac{1}{2}} = 40.$

22. $4u^2 - v^2 = 84,$
$u - 2v = 13.$

23. $u^2 - uv = 104,$
$uv + v^2 = -15.$

24. $\frac{1}{x^2} - \frac{1}{y^2} = 9,$
$\frac{1}{x} - \frac{1}{y} = 9.$

25. $\frac{1}{x^2} - \frac{1}{y^2} = 19,$
$\frac{1}{xy} = 90.$

26. $\frac{1}{x^3} - \frac{1}{y^3} = 152,$
$\frac{1}{x} - \frac{1}{y} = 8.$

27. $x^{-2} - y^{-2} = 20,$
$x^{-1} - y^{-1} = 2.$

28. $\frac{y - x}{xy + 1} = \frac{3}{19}.$
$\frac{x + y}{xy - 1} = \frac{9}{17}.$

29. $2u^2 + v = 3uv - 12,$
$2u - v = 0.$

30. $v + uv = -40,$
$u + uv = -28.$

31. $xy^2 = 64,$
$x^2y = 8.$

32. $3x^2 + 5xy - 2y^2 = 0,$
$x^2 + 2xy = 175.$

33. $x^3 + y^3 = 189,$
$x^2 - xy + y^2 = 21.$

34. $2x^2 - xy - y^2 = 0,$
$9x^2 - 9y^2 = 5.$

35. $x^3 + y^3 = 126,$
$x^2y + xy^2 = 30.$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problem.)

1. Find two numbers whose difference is 5 and whose product is 14.

2. Find two numbers whose sum is 30 and the difference of whose squares is 300.

3. Find two numbers whose product plus their difference is 28 and whose quotient is 5.

4. The area of a right triangle is 120 and its hypotenuse is 26. Find the other two sides.

5. It requires 52 rods of fence to inclose a rectangular lot whose area is one acre. Find the dimensions of the lot.

6. The area of a field containing 9.6 acres is doubled by adding 16 rods to the length and the breadth. Find the dimensions.

7. The area of a rectangular field is 30 acres and one diagonal is 100 rods. Find the dimensions.

8. If a number containing two digits is divided by the product of the digits, the quotient is 2. If the number formed by reversing the digits is divided by their sum, the quotient is 7. Find the number.

9. Two numbers are formed of the same two digits in reverse order. The sum of their squares is 2340. The smaller divided by the larger is $\frac{4}{7}$. Find the numbers.

10. The difference in volume of two cubes is 386 cubic inches and the difference of their edges is 2 inches. Find the edges.

11. A walks 14 miles in $\frac{1}{2}$ hour less time than B. The latter walks $\frac{1}{2}$ mile per hour slower than A. Find the speed of each.

12. The value of a fraction is decreased $\frac{7}{40}$ when 1 is added to the numerator and 3 to the denominator and it is increased $\frac{1}{5}$ if 1 is subtracted from the numerator and 2 from the denominator. Find the fraction.

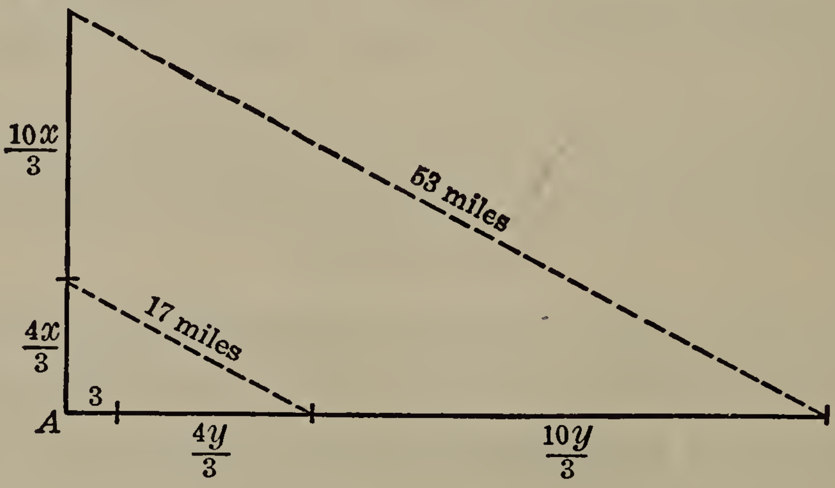
13. If the usual speed of a motor bus is increased 5 miles per hour, 16 minutes will be saved in making a certain run. Twenty-four minutes more will be required on the same run if the rate is decreased 5 miles per hour. Find the length of the run and the usual speed of the bus.

14. The sum of the squares of the digits of a two-digit number is 9 more than the number. Twice the product of the digits equals the number. Find the number.

15. A certain sum at simple interest amounts to $834 in one year. If the rate were $1\frac{3}{4}$ per cent greater and the sum were $100 more, the amount would be $954. Find the sum and the rate.

16. A grocer spent six dollars for oranges. Had they cost four cents a dozen less he would have received five dozen more for the same sum. How many dozen did he buy and what was the price per dozen?

17. A bicyclist leaves *A* and travels north. At the same time a second bicyclist leaves a point 3 miles east of *A* and travels east. One and one-third hours after starting, the shortest distance between them is 17 miles. Three and one-third hours later it is 53 miles. Find the speed of each.



18. Divide 365 into two parts so that they will be squares of consecutive integers.

19. The sum of the radii of two circles is 36 inches and the difference of their areas is $144\,\pi$ square inches. Find the radii.

20. The sum of one number and the reciprocal of another equals 10.25. The sum of the second and the reciprocal of the first equals 4.1. Find the two numbers.

21. A starts out from P to Q at the same time B leaves Q for P. When they meet, A has gone 40 miles more than B. A then finishes the journey to Q in 2 hours and B the journey to P in 8 hours. Find the speeds of A and B and the distance from P to Q.

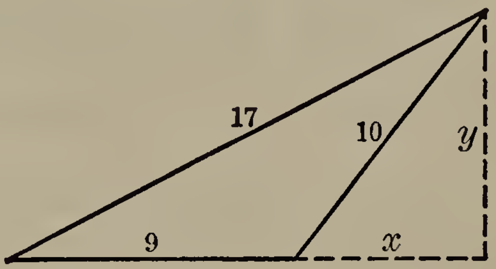
22. A leaves P for Q at the same time that B leaves Q on his way to P. From the time the two meet, it requires $6\frac{2}{3}$ hours for A to reach Q and 15 hours for B to reach P. Find the speed of each if the distance from P to Q is 300 miles.

GEOMETRICAL PROBLEMS

1. The sides of a triangle are 9, 10, and 17. Find the altitude on the side 9.

HINT. From the adjacent figure

$$x^2 + y^2 = 100$$
$$(x + 9)^2 + y^2 = 289$$



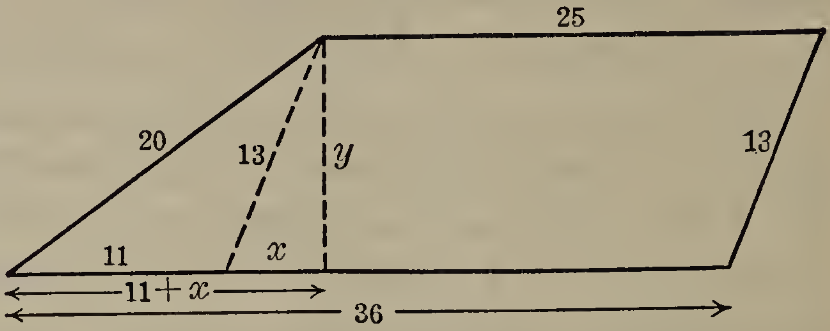
2. The sides of a triangle are 7, 15, 20. Find the altitude on the side 7 and the area of the triangle.

3. What property of a triangle can be used to find the other two altitudes in Exercises 1 and 2? Find the other two altitudes for the triangle in Exercise 2 by this method.

4. The sides of a triangle are 18, 24, and 30. Find the shortest altitude of the triangle.

5. The sides of a triangle are 10, 17, and 21. Find the altitude on the greatest side.

6. The sides of a triangle are 25, 33, and 52. Find the altitude on the side 33.



7. The parallel sides of a trapezoid are 25 and 36 respectively. The nonparallel sides are respectively 13 and 20. Find the altitude of the trapezoid.

HINTS.
$$x^2 + y^2 = 169,$$
$$(11 + x)^2 + y^2 = 400.$$

8. The two parallel sides of a trapezoid are 30 and 51 respectively. The other two sides are 10 and 17 respectively. Find the altitude of the trapezoid and its area.

9. The sides of a trapezoid are 20, 30, 37, and 81 respectively. The sides 30 and 81 are parallel. Find the altitude of the trapezoid and its area.

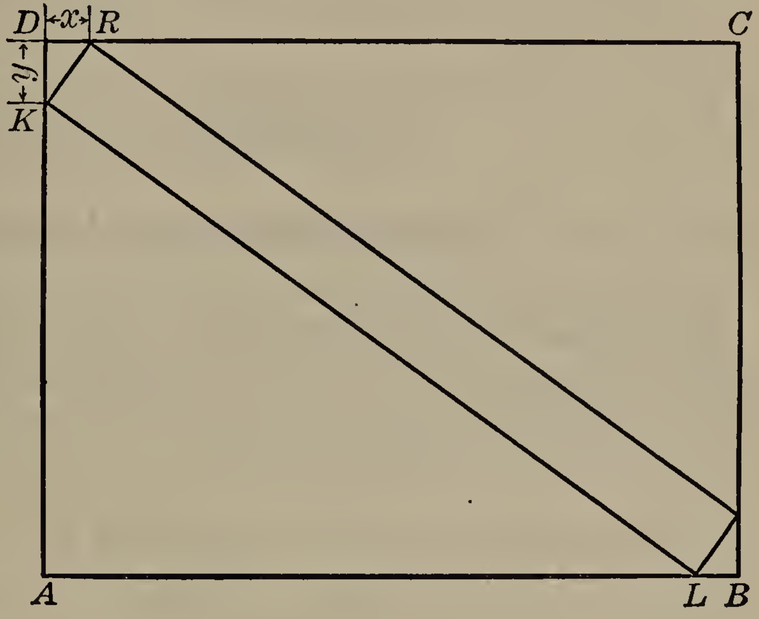
10. The sides of a trapezoid are 14, 18, 40, and 48. The sides 18 and 48 are the bases. Find the altitude and the area of the trapezoid.

11. The sides of a trapezoid are 45, 53, 90, and x. The latter is perpendicular to the sides 45 and 90. Find x and the area of the trapezoid.

12. The hypotenuse of a right triangle is 125 and its area is 2574. Find the other two sides.

13. The area of a right triangle is 96. The difference between the sum of the smaller sides and the hypotenuse is 8. Find the three sides.

14. A rectangular tank is 8 feet 6 inches long and 6 feet 8 inches wide. A board 10 inches wide is laid diagonally on the floor. What two equations must be solved to determine the length of the longest board that can be thus laid?



HINT. Let $DR = x$ and $DK = y$. The triangle DKR is similar to the triangle AKL. If the student desires to find the length of the board, he must solve the system obtained graphically. This is because an attempt to use the other methods presented in this chapter leads to an equation of the fourth degree which cannot be solved by factoring or any simple method. If the student studies advanced algebra, he will learn to solve such equations readily by Horner's method.

CHAPTER XXXIV

THE BINOMIAL THEOREM

228. Powers of binomials. The following identities are easily obtained by actual multiplication:

$$(a+b)^2 = a^2 + 2\,ab + b^2 \tag{1}$$

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3 \tag{2}$$

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4 \tag{3}$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5 \tag{4}$$

If $a + b$ is replaced by $a - b$, the even-numbered terms in each of the preceding expressions will then be negative and the odd-numbered terms will be positive.

229. The expansion of $(a + b)^n$. The form of the expansion for the case where n is an integer is as follows:

The first term is a^n *and the last is* b^n.

The second term is $+ na^{n-1}b$.

The exponents of a *decrease by* 1 *in each term after the first.*

The exponents of b *increase by* 1 *in each term after the second.*

The product of the coefficient in any term and the exponent of a *in that term, divided by the exponent of* b *increased by* 1, *gives the coefficient of the next term.*

The sign of each term is + *if* a *and* b *are positive; the sign of each even-numbered term is* − *if* b *alone is negative.*

According to the above statement we have

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots + b^n.$$

This expresses in symbols the law known as the *binomial theorem.* The theorem holds for all positive values of n and also, with certain limitations, for negative values.

NOTE. The coefficients of the various terms in the binomial expansion are displayed in a most elegant form as follows:

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

In this arrangement each row may be derived from the one above it by observing that each number is equal to the sum of the two numbers, one to the right and the other to the left of it, in the line above. Thus $4 = 1 + 3$, $6 = 3 + 3$, etc. The next line is 1 5 10 10 5 1. The successive lines of this table give the coefficients for the expansions of $(a+b)^n$ for the various values of n. The numbers in the lowest line of the triangle are seen to be the coefficients when $n = 4$; the next line would give those for $n = 5$. This array is known as Pascal's triangle and was published in 1665. It was probably known to Tartaglia nearly a hundred years before its discovery by Pascal.

EXERCISES

Expand by the rule:

1. $(a+b)^6$.
2. $(a-c)^7$.
3. $(a+1)^6$.
4. $(a-1)^7$.
5. $(a+2)^6$.
6. $(a+3)^5$.
7. $(3+a)^5$.
8. $(4-a)^6$.
9. $(5-a)^8$.

Obtain the first four terms of:

10. $(a+b)^{10}$. **12.** $(a+b)^{25}$. **14.** $(a-2)^{18}$.

11. $(a-b)^{15}$. **13.** $(a+1)^{20}$. **15.** $(a-5)^{30}$.

Expand:

16. $(a^2+3c)^5$.

HINTS. To avoid errors in the exponents write

$$(a^2)^5 + \quad (a^2)^4(3c) + \quad (a^2)^3(3c)^2 + \quad (a^2)^2(3c)^3 + \quad (a^2)(3c)^4 + \quad (3c)^5$$

Then in the spaces left for them put in the coefficients of the terms according to the binomial theorem. Finally simplify each term.

17. $(a^3+c)^6$. **18.** $(a^2+3)^7$. **19.** $(a^2-2c^2)^5$.

20. $\left(\frac{a}{c}+c^2\right)^6$. **21.** $\left(\frac{a^2}{c}-\frac{c^3}{a}\right)^4$.

Obtain in simplest form the first four terms of:

22. $(a+5c)^{10}$. **23.** $\left(\frac{a}{c}-c^2\right)^{12}$. **24.** $(a^2-2c)^{20}$.

25. $\left(\frac{a}{c}+\frac{c}{a}\right)^{10}$. **28.** $\left(\frac{a^2}{x}-\frac{x^2}{2a}\right)^{12}$.

26. $\left(\frac{a^2}{c}+c^2\right)^{20}$. **29.** $\left(\frac{6x^3}{a}+\frac{2a^2}{9x}\right)^5$.

27. $\left(\frac{a}{x^2}-\frac{2x}{a^2}\right)^{14}$. **30.** $\left(\frac{\sqrt{a}}{x}-\frac{\sqrt{x}}{a}\right)^{10}$.

31. Write the first six terms of the expansion of $(a+b)^n$, and evaluate it for $n=1$, $n=2$, $n=3$, $n=4$. How does the number of terms compare with n? What is the value of each coefficient after the $(n+1)$th? Why does not the expansion extend to more than five terms when $n=4$?

32. $\left(1 + \frac{1}{n}\right)^n$.

33. $(4\,m^{\frac{3}{2}} - m^{\frac{1}{2}}a^{\frac{1}{3}})^6$.

Compute the following to two decimal places:

34. $(2.1)^8$.

HINT. $(2.1)^8 = (2 + .1)^8$, etc.

35. $(4.2)^{10}$.

36. $(1.04)^7$.

37. $(.97)^{10}$.

HINT. $(.97)^{10} = (1 - .03)^{10}$.

38. $(5.8)^{12}$.

230. Expansion in series. The expansion of section 229 can be proved for small integral values of n by direct multiplication. The theorem holds, however, for any integral value of n. The proof of this fact is not difficult, but it requires a knowledge of mathematical induction which usually comes in an advanced course. For positive integral values of n the number of terms is always $n + 1$. For fractional and negative values of n this is not the case. The reason is that the factors in the coefficients n, $n - 1$, $n - 2$, etc., cannot become zero for fractional or negative values of n, and hence the expansion has no final term and it becomes an infinite series. The expansion is still valid if a is greater than b and is useful for two purposes, the expansion in series and the extraction of roots.

For example

$$(a + b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \tfrac{1}{2}\,a^{-\frac{1}{2}}b - \tfrac{1}{8}\cdot a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}\cdot a^{-\frac{5}{2}}b^3 + \cdots.$$

By giving n the values $\frac{1}{2}$ or $\frac{1}{3}$, one can compute the square root or the cube root of a number to any required degree of accuracy.

In such computations it is desirable to let the number which corresponds to a in the binomial exceed the one corresponding to b by as much as possible and at the same time to have a an integer.

As will be seen, many simple expansions with fractional and negative exponents can be easily verified. That the binomial theorem holds for all fractional and negative values of n when a is greater than b has been proved.

NOTE. The process of extracting the square root and even the cube root by means of the binominal expansion was familiar to the Hindus more than a thousand years ago. The German Stifel (1486–1567) stated the binomial theorem for all powers up to the seventeenth, and also extracted roots of numbers by this method.

EXERCISES

Find the first five terms of:

1. $(1 + x)^{-1}$.

2. Obtain the result of Exercise 1 by actual division.

HINT. $(1 + x)^{-1} = \frac{1}{1 + x}$, etc.

3. $(1 + x)^{-2}$.

4. Obtain the result of Exercise 3 by division.

5. $(28)^{\frac{1}{2}}$.

Solution.

$$\begin{aligned} 28^{\frac{1}{2}} &= (25 + 3)^{\frac{1}{2}} \\ &= (25)^{\frac{1}{2}} + \tfrac{1}{2}(25)^{-\frac{1}{2}}(3) - \tfrac{1}{8}(25)^{-\frac{3}{2}}(3)^2 + \tfrac{1}{16}(25)^{-\frac{5}{2}}(3)^3 \cdots \\ &= 5 + .3 - .009 + .00054 - \cdots = 5.29154. \end{aligned}$$

It is proved in more advanced books that when the terms of an infinite series are alternately plus and minus, and each term is numerically less than the preceding one, the value of the entire sum from a given term on cannot exceed that term. This fact renders these so-called "alternating series" especially convenient for computation, since a definite limit of error is known at each stage of the computation. In this example the error cannot exceed .00054.

6. Check the result of Exercise 5 by extracting the square root by the method of § 64.

7. Show that $(65)^{\frac{1}{3}} = 4.0207$.

Find to three decimals by the binomial theorem:

8. $(26)^{\frac{1}{2}}$. **10.** $(38)^{\frac{1}{2}}$.

9. $(27)^{\frac{1}{2}}$. **11.** $(83)^{\frac{1}{2}}$.

12. $(62)^{\frac{1}{2}}$.

HINT. $(62)^{\frac{1}{2}} = (64 - 2)^{\frac{1}{2}}$.

Here $(64 - 2)^{\frac{1}{2}}$ yields more accurate results with fewer terms than $(49 + 13)^{\frac{1}{2}}$.

13. $(28)^{\frac{1}{3}}$. **14.** $(61)^{\frac{1}{3}}$. **15.** $(29)^{\frac{1}{3}}$.

16. $(520)^{\frac{1}{3}}$. **17.** $(17)^{-\frac{1}{2}}$.

18. Check the result of Exercise 17 by finding the value of $\frac{1}{\sqrt{17}}$ by the ordinary method of square root. Here $\frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}$, etc.

Expand to five terms:

19. $(1 + x)^{\frac{1}{2}}$. **20.** $(2 - x)^{\frac{1}{2}}$. **21.** $(3 + x)^{\frac{1}{3}}$.

22. $(2 + x)^{-\frac{1}{2}}$. **24.** $(1 - x)^{\frac{2}{3}}$.

23. $(8 - x)^{-\frac{1}{3}}$. **25.** $(1 + x)^{-\frac{2}{3}}$.

231. The factorial notation. The notation $5!$ or $\underline{|5}$ signifies $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, or 120. Also $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

In general, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)n$.

The symbol $n!$ or $\underline{|n}$ is read "factorial n."

In the factorial notation the denominators of the fourth and fifth terms of the expansion of $(a+b)^n$ become $3!$ and $4!$ respectively (see formula, p. 519).

EXERCISES

Evaluate:

1. $7!$.
2. $5!\cdot 3!$.
3. $6!\cdot 4!$.
4. $7!\div 4!$.
5. $5!-(4!)^2$.
6. $8!\div 6!$.
7. $n!\div(n-1)!$.
8. $n!\div(n-2)!$.
9. $(2n)!\div(2n-3)!$.
10. $(n+3)!\div(n-1)!$.

11. Evaluate $\dfrac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}$ when $n=6$, $r=4$; $n=12$, $r=9$; $n=20$, $r=14$; $n=18$, $r=12$.

232. The *r*th term of $(a+b)^n$. According to the binomial theorem the fifth term of the expansion on page 519 is

$$+\frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!}.$$

If we note carefully this term and the directions on page 518, we can write down, from the considerations that follow, any required term without writing other terms of the expansion.

The *denominator of the coefficient* of the fifth term is $4!$. From the law of formation the denominator of the sixth term would be $5!$, of the seventh term $6!$, etc. Consequently in the rth term the denominator of the coefficient would be $(r-1)!$.

The *numerator of the coefficient* of the fifth term contains the product of the four factors $n(n-1)(n-2)(n-3)$.

The numerator of the sixth term would contain these four and the additional factor $n-4$. Similarly, the last factor in the numerator of the seventh term would be $n-5$, etc. Hence the last factor in the rth term would be $n-(r-2)$, and the numerator of the coefficient of the rth term is $(n-1)(n-2)(n-3)\cdots(n-r+2)$.

The *exponent of* a in the fifth term is $n-4$, and in the sixth term it would be $n-5$, etc. Therefore in the rth term the exponent of a is $n-(r-1)$, or $n-r+1$.

The *exponent of* b in the fifth term is 4, in the sixth term 5, etc. Therefore in the rth term the exponent of b is $r-1$.

The *sign* of any term of the expansion (if n is a positive integer) is plus if the binomial is $a+b$. If the binomial is $a-b$, the terms containing the odd powers of b will be negative. In other words the sign in such cases depends upon whether the exponent $r-1$ is even or odd, being $+$ if $r-1$ is even, and $-$ if $r-1$ is odd.

Hence the rth term (r not equal to 1) of $(a+b)^n$ equals

$$(-1)^{r-1}\frac{n(n-1)(n-2)(n-3)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}. \quad (1)$$

If we wanted the twelfth term, we would in using (1) substitute 12 for r.

EXERCISES

Find the required term:

1. The sixth term of $(a+b)^{12}$.

Solution. Substituting 12 for n and 6 for r in the formula (1) gives

$$+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2}\cdot a^7b^5$$
$$=+792\,a^7b^5$$

2. Fifth term of $(a + b)^{10}$.
3. Sixth term of $(a + b)^{13}$.
4. Seventh term of $(a - b)^{11}$.
5. Eighth term of $(a - b)^{10}$.
6. Fourth term of $(a^2 - 2x)^{10}$.
7. Fifth term of $\left(a - \frac{1}{a}\right)^{20}$.
8. Sixth term of $\left(\frac{a}{c} - \frac{c}{a}\right)^{14}$.
9. Eighth term of $(x^3 - x^2)^{16}$.
10. Fourth term of $\left(\sqrt{\frac{a}{x}} - \sqrt{x}\right)^{20}$.
11. Find the middle term of $(x^3 - x^2)^{14}$.

Find the coefficient of:

12. x^4 in $(1 + x)^{12}$.
13. x^{10} in $(x^2 + 1)^{20}$.
14. x^{12} in $(x^3 - 1)^{18}$.
15. x^5 in $(x^3 - x^{-2})^{15}$.

NOTE. The binomial theorem occupies a remarkable place in the history of mathematics. By means of it Napier was led to the discovery of logarithms, and its use was of the greatest assistance to Newton in making his most wonderful mathematical discoveries. But to-day the results of Newton and of Napier are explained without even so much as a mention of the binomial theorem, for simpler methods of obtaining these results have been discovered.

It was Newton who first recognized the truth of the theorem, not only for the case where n is a positive integer, which had long been familiar, but for fractional and negative values as well. He did not give a demonstration of the general validity of the binomial development, and none even passably satisfactory was given until that of Euler (1707–1783). The first entirely satisfactory proof of this difficult theorem was given by the brilliant young Norwegian Abel (1802–1829).

CHAPTER XXXV

LOGARITHMS

233. Introduction. Logarithms were invented to shorten the work of extended numeric computations which involve one or more of the operations of multiplication, division, involution, and evolution. Their use has decreased the labor of computing to such an extent that many calculations which would require hours without the use of logarithms can be performed with their aid in a small fraction of that time.

234. Definition of logarithm and base. If we write the equation

$$n = b^l, \tag{1}$$

we express therein the essential relation between a number, n, and its logarithm, l, for a given base, b. In the notation of logarithms this is written

$$\log_b n = l, \tag{2}$$

and it is read " the logarithm of n to the base b equals l."

We can define verbally in one statement both logarithm and base as follows:

The logarithm of a given number is the power to which another number, called the base, must be raised in order to equal the given number.

It is important to realize that equations (1) and (2) are merely two different ways of expressing precisely the same relations, one the exponential way, the other the logarithmic. Above all it is necessary to keep in mind the fact that a logarithm is an exponent.

Thus in $81 = 3^4$, the given number is 81, the base is 3, and the logarithm is 4; that is, $\log_3 81 = 4$.

235. Systems of logarithms. The base of the common, or Briggs, system of logarithms is 10. Hence a table of common logarithms is really a table of exponents of the number 10. Since the greater portion of these exponents are approximate values of irrational numbers, it follows that computations by means of logarithms give only approximate results. Tables exist, however, in which each logarithm is given to twenty or more decimals; hence practically any desired degree of accuracy can be obtained by using the proper table. The common system is used in numerical work almost exclusively. The table on page 534 is a table of common logarithms carried to four decimal places.

The only other system of logarithms used in computations is called the natural system. It has for its base the irrational number 2.7182+, which is usually denoted by the letter *e* and is used mainly for theoretical purposes.

It can be proved that the laws given on pages 401–402, governing the use of rational exponents, hold for irrational exponents. In the work on logarithms this fact will be assumed.

ORAL EXERCISES

1. If $16 = 2^x$, $x = ?$ $\log_2 16 = ?$
2. If $625 = 5^x$, $x = ?$ $\log_5 625 = ?$
3. $\text{Log}_4 64 = ?$ $\log_2 32 = ?$ $\log_3 243 = ?$
4. If $10000 = 10^x$, $x = ?$ $\log_{10} 10000 = ?$

5. $\log_{10} 100 = ?$ $\log_{10} 1000 = ?$ $\log_{10} 10 = ?$ $\log_{10} 1 = ?$

6. $4^0 = ?$ $\log_4 1 = ?$ $\log_6 1 = ?$ $\log_n 1 = ?$

	Number	Base	Logarithm		Number	Base	Logarithm
7.	25	5	?	**22.**	?	10	2
8.	81	3	?	**23.**	?	10	1
9.	81	9	?	**24.**	?	10	0
10.	64	2	?	**25.**	?	10	3
11.	125	5	?	**26.**	?	10	-1
12.	216	6	?	**27.**	?	10	-2
13.	343	7	?	**28.**	?	10	-3
14.	27	?	3	**29.**	4	?	$\frac{1}{2}$
15.	8	?	3	**30.**	8	?	$\frac{1}{3}$
16.	128	?	7	**31.**	27	?	$\frac{3}{2}$
17.	625	?	3	**32.**	5	25	?
18.	1024	?	5	**33.**	6	216	?
19.	?	4	4	**34.**	16	?	$\frac{3}{4}$
20.	?	3	5	**35.**	81	?	$\frac{3}{4}$
21.	?	5	3	**36.**	25	?	$\frac{3}{2}$

Read in the notation of logarithms:

37. $73 = 10^{1.863}$.

38. $50 = 10^{1.69}$.

39. $1 = 10^0$.

40. $.1 = 10^{-1}$.

41. $.2 = 10^{-.699}$.

42. $.089 = 10^{-1.05}$.

43. $1650 = 10^{3.2175}$.

44. $165 = 10^{2.2175}$.

45. $16.5 = 10^{1.2175}$.

46. $1.65 = 10^{.2175}$.

47. $.165 = 10^{-1+.2175}$.

48. $.0165 = 10^{-2+.2175}$.

49. $200 = 10^{2.301}$.

50. $20 = 10^{1.301}$.

Read in exponential notation, with the base 10:

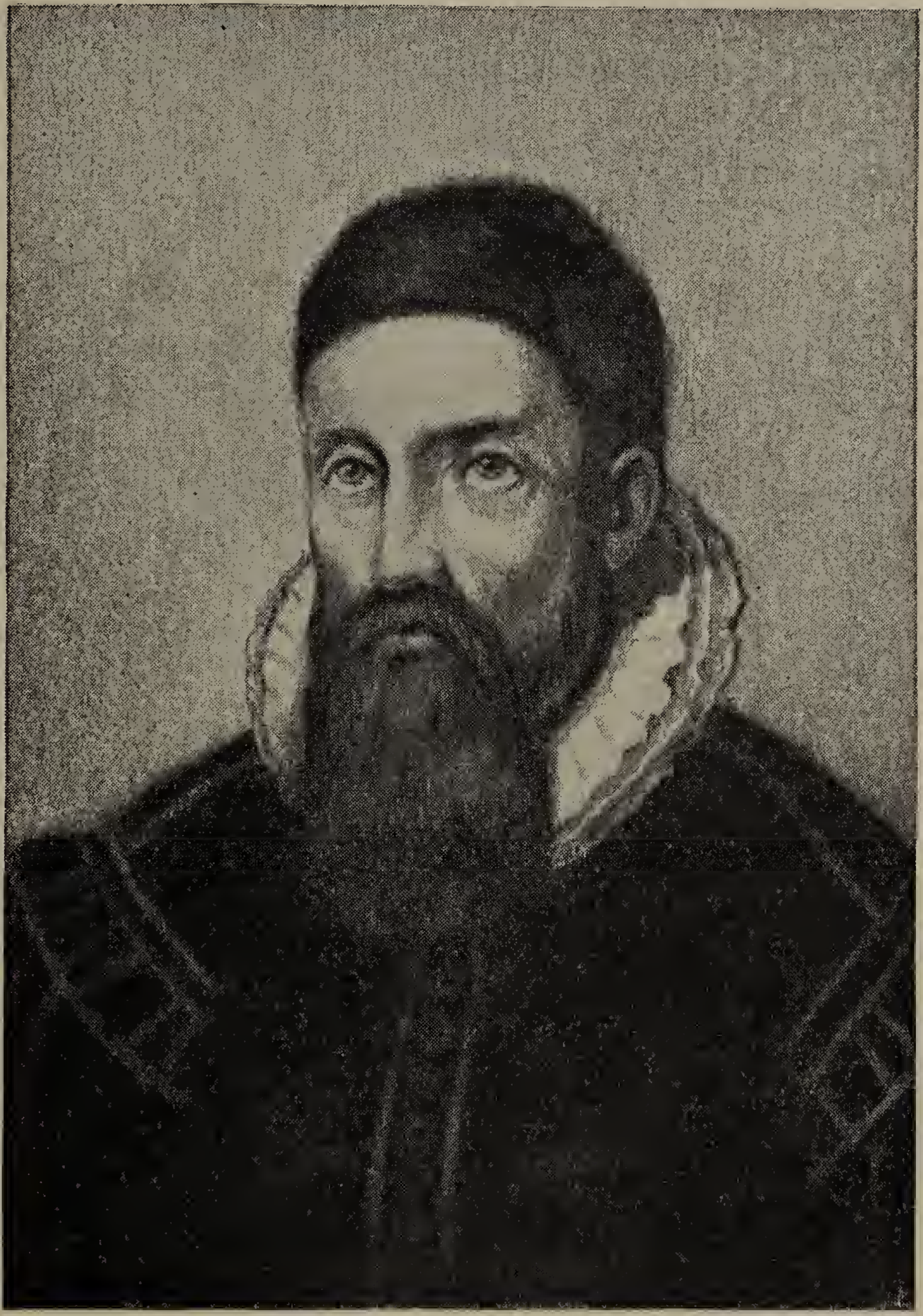
51. $\log 3 = .4771.$
52. $\log 40 = 1.6021.$
53. $\log .0983 = -2 + .9926.$
54. $\log 100^3 = 6.$

Find the numeric value of the following sums:

55. $\log_{10} 10 + \log_{10} 100 + \log_{10} 1000.$
56. $\log_{10} 100 + \log_{10} 1 + \log_{10} .1.$
57. $\log_3 9 + \log_4 16 + \log_5 125.$
58. $\log_7 49 + \log_8 512 + \log_2 4.$
59. $\log_2 8 + \log_3 27 + \log_4 64.$
60. $\log_{36} 6 + \log_{25} 5 + \log_5 25.$

BIOGRAPHICAL NOTE. John Napier. Although many scientists have been honored with titles on account of their discoveries, very few of the titled aristocracy have become distinguished for their mathematical achievements. A notable exception to this rule is found in John Napier, Lord of Merchiston (1550–1617), who devoted most of his life to the problem of simplifying arithmetic operations. Napier was a man of wide intellectual interests and great activity. In connection with the management of his estate he applied himself most seriously to the study of agriculture, and experimented with various kinds of fertilizers in a somewhat scientific manner, in order to find the most effective means of reclaiming soil. He spent several years in theological writing. When the danger of an invasion by Philip of Spain was imminent, he invented several devices of war. Among these were powerful burning mirrors, and a sort of round musket-proof chariot, the motion of which was controlled by those within, and from which guns could be discharged through little portholes.

But by far the most serious activity of Napier's life was the effort to shorten the more tedious arithmetic operations. He invented the first approximation to a computing machine, and also devised a set of rods, often called Napier's bones, which were of assistance in multiplication. His crowning achievement, however, was the invention of logarithms, to which he devoted fully twenty years of his life.



John Napier

236. Steps preceding computation. Before computation by means of the table can be taken up, two processes requiring considerable explanation and practice must be mastered.

I. *To find from the table the logarithm of a given number.*

II. *To find from the table the number corresponding to a given logarithm.*

237. Characteristic and mantissa. Unless a number is an exact power of 10, its logarithm consists of an integer and a decimal.

This fact is illustrated in Exercises 37–50, page 529.

The integral part of a logarithm is called its characteristic.

The decimal part of a logarithm is called its mantissa.

Log 800 = 2.903. Here 2 is the characteristic and .903 is the mantissa.

The characteristic of any number is obtained not from a table of logarithms but by an inspection of the number itself, according to rules which will now be derived.

$10^4 = 10{,}000$; that is, the log $10{,}000 = 4$.
$10^3 = 1000$; that is, the log $1000 = 3$.
$10^2 = 100$; that is, the log $100 = 2$.
$10^1 = 10$; that is, the log $10 = 1$.
$10^0 = 1$; that is, the log $1 = 0$.
$10^{-1} = .1$; that is, log $.1 = -1$.
$10^{-2} = .01$; that is, the log $.01 = -2$.
$10^{-3} = .001$; that is, the log $.001 = -3$.

The preceding statement indicates between what two integers the logarithm of a number less than 10,000 lies. This determines the characteristic.

Since 347 lies between 100 and 1000 (that is, between 10^2 and 10^3), log 347 must lie between 2 and 3 and must equal 2 (characteristic) plus a decimal (mantissa). Mantissas are always obtained from a table like that on page 244.

And since .0037 lies between .001 and .01 (that is, 10^{-3} and 10^{-2}), log $.0037 = -3$ plus a positive decimal or -2 plus a negative decimal.

For the determination of the characteristic of a positive number we have the following rules:

I. *The characteristic of a number greater than* 1 *is one less than the number of digits to the left of the decimal point.*

II. *The characteristic of a number less than* 1 *is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.*

Accordingly the characteristic of 2874 is 3; of 8 is 0; of .3 is -1; of .046 is -2; of .0078 is -3.

ORAL EXERCISES

What is the characteristic of:

1. 418.
2. 7863.
3. 79.
4. 8.
5. 7300.
6. 730.
7. 73.
8. 7.3.
9. .73.
10. .073.
11. .0073.
12. .00073.
13. 71.067.
14. .000091.
15. .60000.
16. .348.

The table on pages 534–535 gives the mantissa of numbers from 100 to 999. Before each mantissa a decimal point is understood.

The numbers 7860, 786, 78.6, 7.86, .0786, and .00786 are spoken of as composed of the same significant digits in the same order. They differ only in the position of the decimal point, and consequently their logarithms to the base 10 will have different characteristics, but they will have the same mantissa.

The last two points are easily illustrated by any two numbers which have the same significant digits in the same order.

$$8.26 = 10^{.917}, \text{ that is, } \log 8.26 = .917$$
$$8.26 \cdot 10^2 = 826 = 10^{.917} \cdot 10^2 = 10^{2.917}$$
$$\text{or } \log 826 = 2.917.$$

The property just explained does not belong to a system of logarithms in which the base is any number other than 10. Thus, if the base is 100, the most convenient number after 10, the logarithms of 7860, 786, 78.6, and 7.86 are respectively 1.9477, 1.4477, .9477, and .4477. While a certain regularity in characteristic and mantissa can be seen here, it is obvious that the rules for obtaining them would not be so simple as they are for the base 10. Moreover, it can be seen from the illustration just given that tables of a given accuracy are far shorter with the base 10 than they would be with any other base.

238. Use of the table. The table on page 534 gives the mantissas of all numbers containing one, two, or three significant digits. To obtain the logarithms of a number containing three or fewer significant digits we have the

RULE. *Determine the characteristic by inspection. Find in column "N" the first two significant figures of the given number. In the row with these and in the column headed by the third figure of the number, find the required mantissa.*

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

ORAL EXERCISES

Find the logarithm of:

1. 74.
2. 381.
3. 485.
4. 739.
5. 63.
6. 21.
7. 5.
8. 50.
9. 500.
10. 97.
11. 738.
12. .487.

Solution. The characteristic of .487 is -1 and the mantissa is .6875. Hence $\log .487 = -1 + .6875$. This is often written in the abbreviated form, $\bar{1}.6875$. The mantissa is always kept positive in order to avoid the addition and subtraction of both positive and negative decimals, which in ordinary practice contain from three to five figures. Negative characteristics, being integers, are comparatively easy to take care of. (The student should note that log .487 is really negative, being $-1 + .6875$, or $-.3125$.)

13. .638.
14. .0782.
15. .00964.
16. .0307.
17. 63800.
18. 27000.
19. 3.75.
20. .784.

239. Interpolation. If a number contains four or more significant digits, its mantissa is not given in the table on page 534. It is possible, however, to calculate the mantissas of four- and five-digit numbers from mantissas which are given in the table. This process or the reverse of it is called interpolation.

If we desire the logarithm of a number not in the table, say 3575, we know that it lies between the logarithms of 3570 and 3580, which are given in the table. Since 3575 is halfway between 3570 and 3580, we assume, though it is only approximately true, that the required logarithm is halfway between their logarithms, 3.5527 and 3.5539. In order to find log 3575 we first look up log 3570 and log 3580 and then take half (or .5) their difference (this difference may usually be taken from the column headed D) and add it to log 3570. This gives

$$\log 3575 = 3.5527 + .5 \times .0012 = 3.5533$$

Were we finding log 3578 we should take .8 of the difference between log 3570 and log 3580 and add it to log 3570 as follows:

$$\begin{aligned}\log 3578 &= 3.5527 + .8 \times .0012\\ &= 3.5527 + .00096\\ &= 3.5527 + .0010\\ &= 3.5537\end{aligned}$$

Observe that in using four-place tables one should not carry results to five figures. If the fifth figure of the logarithm is 5, 6, 7, 8, or 9, omit it and increase the fourth figure by 1; that is, *obtain results to the nearest figure in the fourth place.*

For finding the logarithm of a number not given in the table we have the

Rule. *Prefix the proper characteristic to the mantissa of the first three significant figures.*

Then multiply the difference between this mantissa and the next greater mantissa in the table (*called the tabular difference*) *by the remaining figures of the number preceded by a decimal point.*

Add the product to the logarithm of the first three figures, taking the nearest decimal in the fourth place.

In this method of interpolation we have assumed that the increase in the logarithm is directly proportional to the increase in the number. As has been said, this is not strictly true, yet the results thus obtained are nearly always correct to the fourth decimal place.

EXERCISES

Find the logarithm of:

1. 4785.
2. 6334.
3. 42.67.
4. 96.42.
5. 3.528.
6. 981.6.
7. 42.83.
8. 8731.
9. 46.209.
10. .01877.
11. .0004444.
12. .034567.
13. 3.1416.
14. 2.1782.
15. 3468000.
16. 19635.

240. Antilogarithms. An antilogarithm is the number corresponding to a given logarithm. Thus antilog 3 equals 1000.

If we desire the antilogarithm of a given logarithm, say 4.5132, we proceed as follows: The mantissa .5132 is found in the row which has 32 in column N and in the column which has 6 at the top. Hence the first three significant figures of the antilogarithm are 326. Since the characteristic is 4, the number must have five digits to the left of the decimal point. Thus antilog 4.5132 = 32600.

Therefore, if the mantissa of a given logarithm is found in the table, its antilogarithm is obtained by the

Rule. *Find the row and the column in which the given mantissa lies. In the row found take the two figures which are in column "N" for the first two significant figures of the antilogarithm and for the third figure the number at the top of the column in which the mantissa stands.*

Place the decimal point as indicated by the characteristic.

ORAL EXERCISES

Find the antilogarithm of:

1. 2.9304.
2. 1.7839.
3. .5635.
4. $\bar{1}.6284$.
5. $\bar{2}.4502$.
6. 5.8904.
7. 8.6222 − 10.

HINT. 8.6222 − 10 = − 2 + .6222.

8. 7.8261 − 10.
9. 6.5877.
10. 4.1335 − 7.
11. − 3 + .6304.
12. .9053.
13. 7.9961 − 10.
14. 9.7443 − 10.

If the mantissa of a given logarithm, as in 1.4571, is not given in the table, the antilogarithm is obtained by interpolation as follows:

The mantissa .4571 lies between

.4564, the mantissa of the sequence 286,

and .4579, the mantissa of the sequence 287.

Therefore the antilogarithm of 1.4571 lies between 28.6 and 28.7. Since the tabular difference is 15 and the difference between .4564 and .4571 is 7, the mantissa .4571 lies $\frac{7}{15}$ of the way from .4564 to .4579. Therefore the required antilogarithm lies $\frac{7}{15}$ of the way from 28.6 to 28.7.

Then $\text{antilog } 1.4571 = 28.6 + \frac{7}{15}(.1)$

and $28.6 + .046 = 28.65.$

Therefore to find the number corresponding to a given mantissa when the mantissa is not found in the table we have the

Rule. *Write the number of three figures corresponding to the lesser of two mantissas between which the given mantissa lies.*

Subtract the less mantissa from the given one and divide the remainder by the tabular difference to two decimal places. If the second digit is 5 *or more, increase the first digit by* 1; *if less than* 5, *omit it.*

Annex the resulting digit to the three already found and place the decimal point where indicated by the characteristic.

EXERCISES

Find the antilogarithms of:

1. 1.4860.
2. 2.4796.
3. 4.9481.
4. 0.3727.
5. $\bar{1}.3626$.
6. 9.8448 – 10.
7. 6.0748 – 9.
8. 5.8153 – 10.
9. .6187.
10. 7.5257 – 10.
11. 8.4230.
12. 8.6510 – 10.

241. Multiplication. Multiplication by logarithms depends on the

Theorem. *The logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

That is, for the numbers x and z

$$\log_b(x \cdot z) = \log_b x + \log_b z.$$

Proof. Let $\log_b x = l_1$ (1)

and $\log_b z = l_2$. (2)

From (1), $x = b^{l_1}$. (3)

From (2), $z = b^{l_2}$. (4)

(3) × (4), $xz = b^{l_1 + l_2}$. (5)

Therefore,

$$\begin{aligned} \log_b xz &= l_1 + l_2 \\ &= \log_b x + \log_b z. \end{aligned}$$

It follows from this theorem that the logarithm of the product of more than two numbers is the sum of the logarithms of the factors. Thus $\log(x \cdot y \cdot z) = \log x + \log y + \log z$.

EXERCISES

Perform the indicated operation by logarithms:

1. 16 × 35.

Solution.
$$\begin{aligned} \log 16 &= 1.2041 \\ \underline{\log 35} &\underline{= 1.5441} \end{aligned}$$

Adding, $\log(16 \times 35) = 2.7482$

Antilog $2.7482 = 560$

2. 27 × 38.

3. 65 × 79.

4. 318 × 7.

5. 7.6 × 4.91.

6. 231 × 47.

7. 873 × 67.

8. 769 × 257.

9. 7860 × 1325.

10. 64.37 × 3.142.

11. 2600 × 318.

12. 3718 × 5694.

13. 3470 × .0485.

Solution.
$$\begin{aligned} \log(3470 \times .0485) &= 3.5403 + 8.6857 - 10 \\ &= 12.2260 - 10 = 2.2260 \end{aligned}$$

Antilog $2.2260 = 168.3$.

Since the mantissa is always positive, any number carried over from the tenths' column to the units' column is positive. This occurs in the preceding solution, where $.6 + .5 = 1.^{+}$, giving $+1$ to be added to the sum of the characteristics $+3$ and -2, in the units' column. Mistakes in such cases will be less frequent if the logarithms with negative characteristics be written as in the 8–10 notation.

14. 438 × .725.

15. 637 × .0354.

16. 91.7 × .00637.

17. .00842 × .6071.

18. (4316)(– 3.81)

HINT. Determine by inspection the sign of the product. Then calculate as if all signs were positive and give to the result the proper sign.

19. 23(– 284).

20. $4.2 \times 96 \times 7.1$.

21. $4.83 \times 2.36 \times .971$.

22. 57(– 28)(– 3.65).

242. Division. Division by logarithms depends on the

THEOREM. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

That is, for the numbers x and z

$$\log_b \frac{x}{z} = \log_b x - \log_b z.$$

Proof. Let $\log_b x = l_1$ (1)

and $\log_b z = l_2$. (2)

From (1), $x = b^{l_1}$. (3)

From (2), $z = b^{l_2}$. (4)

(3) ÷ (4), $\frac{x}{z} = b^{l_1 - l_2}$.

Therefore,

$$\log_b \frac{x}{z} = l_1 - l_2$$
$$= \log_b x - \log_b z.$$

It follows from the multiplication and the division theorems that if the product of two or more numbers is divided by the product of two or more others, the logarithm of the resulting quotient is the sum of the logarithms of the factors of the dividend minus the sum of the logarithms of the factors of the divisor. Thus

$$\log \frac{abc}{xyz} = \log a + \log b + \log c - (\log x + \log y + \log z).$$

EXERCISES

Using logarithms, perform the indicated operations:

1. $765 \div 34$.

Solution.

$$\begin{array}{rl} \log 765 &= 2.8837 \\ \log 34 &= 1.5315 \\ \hline \text{Subtracting,}\quad \log (765 \div 34) &= 1.3522 \\ \text{antilog } 1.3522 &= 22.5 \end{array}$$

2. $625 \div 25$.

3. $784 \div 37$.

4. $563 \div 4.27$.

5. $4380 \div 74.3$.

6. $79.43 \div 3.25$.

7. $468.4 \div 47.2$.

8. $3754 \div 2869$.

9. $4.38 \div .0735$.

Solution.

$$\begin{array}{rlr} \log 4.38 = & .6415 = & 10.6415 - 10 \\ \log .0735 = & & 8.8663 - 10 \\ \hline \log (4.38 \div .0735) = & & 1.7752 \\ \text{Antilog} & 1.7752 = 59.6 & \end{array}$$

10. $3.76 \div .0759$.

11. $4.23 \div .687$.

12. $25.37 \div .00786$.

13. $.738 \div .0381$.

14. $.0156 \div .492$.

15. $.0864 \div 73.52$.

16. $.0693 \div 42.86$.

17. $43 \times 728 \div 35.24$.

18. $87.2 \times 43.9 \div 28.46$.

19. $\dfrac{468 \times 75.34}{381 \times 27}$.

20. $\dfrac{497(.7852)}{3.142}$.

21. $\dfrac{439(-8.64)3056}{(-75.3)(-8.61)}$.

243. Raising to a power by means of logarithms. Raising to a power by means of logarithms depends on the

THEOREM. *The logarithm of the* r*th power of a number is* r *times the logarithm of the number.*

That is, for the numbers r and x, $\log_b x^r = r \log_b x$.

Proof. Let $\log_b x = l.$ (1)

Then $x = b^l.$ (2)

Raising both members of (2) to the rth power,

$x^r = b^{rl}.$ (3)

Therefore $\log_b x^r = rl.$ (4)

From (1) and (4) $\log_b x^r, = r \log_b x.$

EXERCISES

Compute, using logarithms:

1. $(3.42)^4$.

Solution. $\log 3.42 = .5340.$

$\log (3.42)^4 = 4(.5340) = 2.1360.$

Therefore $(3.42)^4 = \text{antilog } 2.1360 = 136.8.$

2. $(4.59)^3$.
3. $(47.91)^2$.
4. $(3.812)^4$.
5. $(.0738)^3$.

Solution. $\log .0738 = \bar{2}.8681 = 8.8681 - 10$

$\log (.0738)^3 = 3(\bar{2}.8681) = 3(8.8681 - 10)$

$= \bar{4}.6043 = 26.6043 - 30.$

Therefore $(.0738)^3 = \text{antilog } \bar{4}.6043 = .0004021.$

6. $(.0874)^4$.
7. $(.007495)^2$.
8. $(.9284)^3$.
9. $6984(842)^2$.
10. $482(31.54)^2$.
11. $\dfrac{(43)^2(8381)^3}{(.097)^2(435)^4}$.
12. $.967 \div (.0723)^3$.

244. Extracting a root by means of logarithms. Extracting a root by means of logarithms depends on the

Theorem. *The logarithm of the real* r*th root of a number is the logarithm of the number divided by* r.

That is, for the real numbers r and n, $\log_b \sqrt[r]{n} = \frac{1}{r}\log_b n$.

Proof. Let $\log_b n = l$. (1)

Then $n = b^l$. (2)

Extracting the rth root of both members of (2),

$$(n)^{\frac{1}{r}} = (b^l)^{\frac{1}{r}} = b^{\frac{l}{r}}. \tag{3}$$

Therefore $\log_b (n)^{\frac{1}{r}} = \frac{l}{r}\cdot$

EXERCISES

Compute, using logarithms:

1. $\sqrt[3]{495}$.

Solution. $\log 495 = 2.6946$.

$\log \sqrt[3]{495} = \frac{1}{3}(2.6946) = .8982$.

Therefore $\sqrt[3]{495} = \text{antilog } .8982 = 7.91$.

2. $\sqrt[3]{876}$.
3. $\sqrt[3]{1925}$.
4. $\sqrt[4]{1732}$.
5. $\sqrt[3]{.0785}$.

Solution. $\log .0785 = \bar{2}.8949$.

If one divided $\bar{2}.8949$ as it stands by 3, he would be likely to confuse the negative characteristic and the positive mantissa. This and other difficulties may be easily avoided by adding to the characteristic and subtracting from the resulting logarithm any integral multiple of the index of the root which will make the characteristic positive.

Thus $\log .0785 = 7.8949 - 9$.

Dividing by 3,

$\log \sqrt[3]{.0785} = 2.6316 - 3$.

Therefore $\sqrt[3]{.0785} = \text{antilog } \bar{1}.6316 = .4282$.

6. $\sqrt{.0853}$.

7. $\sqrt[3]{.0007625}$.

8. $\sqrt[4]{.005937}$.

9. $\sqrt[5]{.7648}$.

10. $(5.37)^{\frac{2}{3}}$.

11. $(5.368)^{\frac{3}{2}}$.

12. $(-7.953)^{\frac{4}{3}}$.

13. $\sqrt{49.5(7.38)^2}$.

14. $\sqrt[3]{(.0395)^2}$.

15. $\sqrt[7]{361}$.

16. $\sqrt{97}\sqrt[3]{742}$.

17. $\frac{32}{518} \cdot \frac{\sqrt{431}}{217} \sqrt[3]{792}$.

18. Determine the logarithms of 4267, 426.7, 42.67, and 4.267 to the base 10 and to the base 100. Compare the results. What fact about logarithms do these results emphasize?

NOTE. The preceding four-place table will usually give results correct to one half of one per cent. Five-place tables give the mantissa to five decimal places of the numbers from 1 to 9999 and, by interpolation, the mantissa of numbers from 1 to 99,999. Such tables give results correct to one twentieth of one per cent, a degree of accuracy which is sufficient for most engineering work.

Six-place tables give the mantissa to six decimals for the same range of numbers as a five-place table, but the labor of using a six-place table is much greater than that of using a five-place one.

Seven-place tables contain the mantissas of the numbers from 1 to 99,999. Such tables are needed in certain kinds of engineering work and are of constant use in astronomy.

In place of a table of logarithms engineers often use an instrument called a slide rule. This is really a mechanical table of logarithms arranged ingeniously for rapid practical use. Results can be obtained with such an instrument far more quickly than with an ordinary table of logarithms, and that without recording or even thinking of a single logarithm. A slide rule ten inches long usually gives results correct to three figures. For work requiring greater precision larger and more elaborate instruments which give a ten-figure accuracy are used.

245. Exponential equations. An exponential equation is an equation in which the unknown occurs in an exponent.

Many exponential equations are readily solved by means of logarithms, since $\log a^x = x \log a$.

Thus let $a^x = c$. Then $x \log a = \log c$. Whence $x = \log c \div \log a$.

EXAMPLE

Solve for x: $7^x = 283$.

Solution. $\log 7^x = \log 283$

or $x \log 7 = \log 283$.

Whence $$x = \frac{\log 283}{\log 7} = \frac{2.4518}{.8451} = 2.901.$$

EXERCISES

Solve for x:

1. $4^x = 31$. 2. $7^x = 5$. 3. $18^{2x} = 43$.

4. $9^{x+1} = 43$. 6. $5^{2x-1} = 81$.

5. $3 = 1.05^x$. 7. $(.75)^x = 52$.

Solution. $x \log .75 = \log 52$.

$$x = \frac{\log 52}{\log .75} = \frac{1.7160}{\bar{1}.8751} = \frac{1.7160}{-1+.8751} = \frac{1.7160}{-.1249} = -13.74.$$

8. $(.0863)^x = .284$. 10. $(.0073)^{x+1} = (.084)^2$.

9. $(242)^{1-x} = 500$. 11. $(72)^{2x-3} = (28)^x$.

MISCELLANEOUS EXERCISES

1. Prove $\log ab = \log a + \log b$.

2. Prove $\log \frac{a}{b} = \log a - \log b$.

3. Prove $\log (a)^n = n \log a$.

4. Prove $\log \sqrt[n]{a} = \frac{1}{n} \log a$.

5. Does the logarithm of a negative number to a positive base exist? Explain.

Find, without using the table, the numeric values of:

6. $\log_2 4$.

7. $\log_5 25$.

8. $2 \log_3 27$.

9. $\log_9 27$.

10. $4 \log_5 125$.

11. $\log_4 8 + \log_8 4$.

12. $\log_8 16 + 4 \log_{16} 8$.

13. $4 \log_{27} 81 + 5 \log_{81} 27$.

14. $4 \log_5 125 - 3 \log_{25} 5 + 2 \log_5 25$.

15. $5 \log_2 (\frac{1}{2}) + 3 \log_4 (\frac{1}{64}) + 7 \log_8 4$.

Simplify:

16. $\log \frac{4}{9} + \log \frac{27}{48}$.

17. $\log \frac{17}{27} - \log \frac{34}{63}$.

18. $\log \frac{35}{9} + \log \frac{15}{14} - \log \frac{6}{25}$.

19. $3 \log 5 + 5 \log 3$.

20. $2 \log 9 + 3 \log 5 - 7 \log 15$.

Show that

21. $\log \sqrt{a^2 - x^2} = \frac{1}{2} \log (a + x) + \frac{1}{2} \log (a - x)$.

22. $\log \dfrac{a^3 + x^3}{a} = \log (a + x) + \log (a^2 - ax + x^2) - \log a$.

23. $\log \sqrt{\dfrac{s(s - b)(s - c)}{s - a}} =$

$\frac{1}{2}[\log s + \log (s - b) + \log (s - c) - \log (s - a)]$.

In the following obtain results to four figures:

24. The circumference of a circle is $2 \pi R$. ($\pi = 3.1416$.) Find the circumference of a circle whose radius is 23.15.

25. The area of a circle is πR^2. Find the area of a circle whose diameter is 18.76 inches.

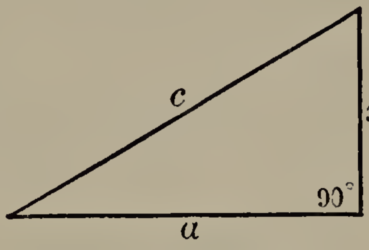
26. The volume of a sphere is $\frac{4}{3}\pi R^3$. Find the volume of a sphere whose diameter is .7854.

27. The area of the surface of a sphere is $4\pi R^2$. Find the area of the earth's surface, the radius being 3959 miles.

If two sides of a right triangle are given and one of the two is the hypotenuse, the other side can be computed by logarithms. Let c be the hypotenuse and a the other given side, then if x is the unknown side, $x^2 = c^2 - a^2$

or $x = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)}$

or $\log x = \frac{1}{2}\log(c+a) + \frac{1}{2}\log(c-a)$.



28. The hypotenuse of a right triangle is 435 and one other side is 347. Find the third side.

29. Find the number of digits in: (*a*) 2^{43}; (*b*) $(3)^{200}$.

30. In how many years will one dollar double itself at four per cent compound interest?

Solution. At the end of one year the amount of one dollar at four per cent is one dollar and four cents; at the end of two years it is (1.04)(1.04), or $(1.04)^2$; at the end of three years it is $(1.04)^3$, and at the end of x years it is $(1.04)^x$.

If x is the number of years required, $(1.04)^x = 2$.

Taking the logarithms of both members of the equation,

$$x \log 1.04 = \log 2$$

Solving, $$x = \frac{\log 2}{\log 1.04} = \frac{.3010}{.0170} = 17.70.$$

In making computations of this nature by the aid of logarithms, care must be exercised not to retain more significant figures in the result than are given with accuracy by the process.

31. In how many years will one dollar double itself at five per cent interest compounded annually?

32. In how many years will any sum of money treble itself at four and one fourth per cent interest compounded annually?

33. About 300 years ago the Dutch paid twenty-four dollars for the island of Manhattan. At four per cent interest compounded annually, what would this payment amount to at the present time?

34. In how many years will one dollar double itself at four per cent interest compounded semi-annually?

35. What will two thousand five hundred dollars amount to in 12 years at 4 per cent interest compounded quarterly?

36. Show that the amount of P dollars in t years at r per cent interest compounded annually is $P(1+r)^t$; compounded semi-annually is $P\left(1+\frac{r}{2}\right)^{2t}$; compounded quarterly is $P\left(1+\frac{r}{4}\right)^{4t}$; and compounded monthly is $P\left(1+\frac{r}{12}\right)^{12t}$.

Solve for x:

37. $a^x = m^{x-1}$.

38. $a^{x+1} = m^{x-2}$.

39. $a^{x+1} \cdot m^x = c^{3x}$.

40. $A = P(1+r)^x$.

41. $m^{x^2+2x} = n$.

42. $3^x \cdot 2^{\frac{1}{x}} = 6$.

43. $a^{4x} + 8\,a^{2x} = 6\,a^{3x}$.

44. $a^{5x} + a^{4x} = 6\,a^{4x} - 6\,a^{3x}$.

45. Show that $a^{\log_a x} = x$.

46. Show that $e^{\log_e (x^2-3x)} = x^2 - 3\,x$.

Note. It is not a little remarkable that just at the time when Galileo and Kepler were turning their attention to the laborious computation of the orbits of planets, Napier should be devising a method which simplifies these processes. It was said a hundred years ago, before astronomical computations became so complex as they now are, that the invention of logarithms, by shortening the labors, doubled the effective life of the astronomer. To-day the remark is well inside the truth.

In the presentation of the subject in modern textbooks a logarithm is defined as an exponent. But it was not from this point of view that they were first considered by Napier. In fact it was not till long after his time that the theory of exponents was understood clearly enough to admit of such application. This relation was noticed by the mathematician Euler, about one hundred and fifty years after logarithms were invented.

It was by a comparison of the terms of certain arithmetical and geometric progressions that Napier derived his logarithms. They were not exactly like those used commonly to-day, for the base which Napier used was not 10. Soon after the publication (1614) of Napier's work, Henry Briggs, an English professor, was so much impressed with its importance that he journeyed to Scotland to confer with Napier about the discovery. It is probable that they both saw the necessity of constructing a table for the base 10, and to this enormous task Briggs applied himself. With the exception of one gap, which was filled in by another computer, Briggs's tables form the basis for all the common logarithms which have appeared from that day to this.

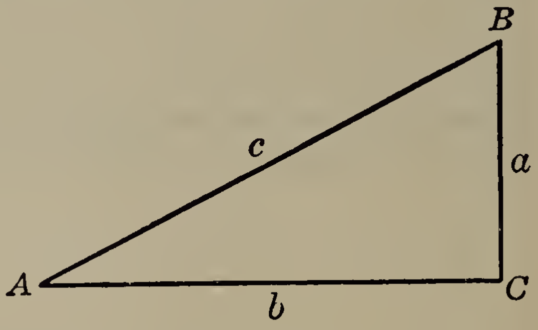
CHAPTER XXXVI

TRIGONOMETRY

246. Introduction. In surveying, astronomy, and in countless other applications of science it is necessary to solve triangles. A triangle has six parts, three sides and three angles. When three of these parts, of which one at least must be a side, have been given or have been measured, the other three can be computed by the methods of trigonometry. In developing the principles of trigonometry it is simplest to begin with the right triangle.

In order to attack the problem stated above it is necessary to denote the ratios of the sides of a right triangle by certain names.

Let ABC be any right triangle with the right angle at C.



$$\text{The sine of the angle } A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}.$$

$$\text{The cosine of the angle } A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}.$$

$$\text{The tangent of the angle } A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}.$$

$$\text{The cotangent of angle } A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}.$$

These definitions are abbreviated as follows:

$$\sin A = \frac{a}{c},\quad \cos A = \frac{b}{c},\quad \tan A = \frac{a}{b},\quad \cot A = \frac{b}{a}.$$

The values of the trigonometric ratios for all angles have been computed to many decimal places. The table on page 555 gives the values of the sine, the cosine, and the tangent correct to four decimal places for every degree from zero to ninety degrees.

EXERCISES

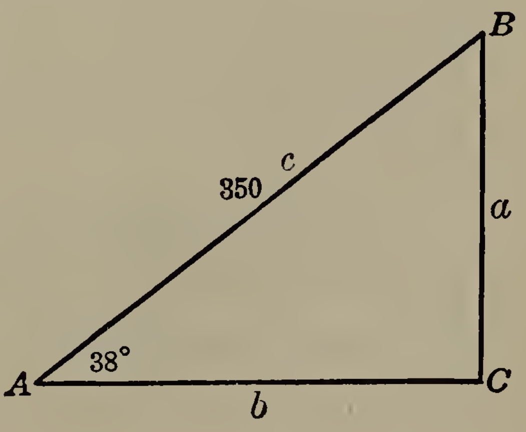
1. Read from the table:

sin 31° sin 68° cos 18° cos 72°
tan 27° tan 82° cot 40° cot 73°

2. From the table determine the number of degrees in angle x if:

(*a*) $\sin x = .9962$.
(*b*) $\sin x = .5150$.
(*c*) $\cos x = .9135$.
(*d*) $\cos x = .4848$.
(*e*) $\tan x = 3.7321$.
(*f*) $\tan x = .5774$.
(*g*) $\cot x = 1.8807$.
(*h*) $\cot x = .6009$.
(*i*) $\cos x = .8572$.
(*j*) $\sin x = .9994$.
(*k*) $\cot x = 57.290$.

3. In the right triangle ABC, side $c = 350$ and angle $A = 38°$. Find the other parts.



HINT. $\sin A = \dfrac{a}{c}.$

$$\sin 38° = \frac{a}{350}.$$

$$a = 350(\sin 38°)$$
$$= 350(.6157) = 215.5.$$

Find the missing parts in the following right triangles. Perform the multiplications and divisions as in Exercise 3 above, or by logarithms if the teacher prefers.

	a	b	c	A	B
4.	?	?	300	18°	?
5.	120	?	?	?	56°
6.	?	225	?	67°	?
7.	?	86.6	100	?	?
8.	707.1	?	1000	?	?
9.	40	20.38	?	?	?
10.	376.8	500	?	?	?
11.	17.05	?	25	?	?
12.	?	73.08	80	?	?
13.	606	1500	?	?	?
14.	125	50.5	?	?	?
15.	?	67.84	80	?	?
16.	1386	?	?	78°	?
17.	?	214	?	?	53°
18.	10.45	99.45	?	?	?

247. Interpolation. The word "interpolation" means literally a placing between. The process consists in finding from the numbers in a given table a number not listed there. It applies to any table, one of square roots, cube roots, or logarithms, or a trigonometric table, such as that given on page 555.

There are two problems of interpolation in a trigonometric table:

(*a*) *Given an* **angle** *not in the table, to find the corresponding sine, cosine, tangent, or cotangent.*

(*b*) *Given the* **value** *of a sine, cosine, tangent, or cotangent not in the table, to find the corresponding angle.*

Four-place Trigonometric Table

Angle	sin	cos	tan	cot	Angle	sin	cos	tan	cot
0°	.0000	1.0000	.0000	∞	45°	.7071	.7071	1.0000	1.0000
1°	.0175	.9998	.0175	57.290	46°	.7193	.6947	1.0355	.9657
2°	.0349	.9994	.0349	28.636	47°	.7314	.6820	1.0724	.9325
3°	.0523	.9986	.0524	19.081	48°	.7431	.6691	1.1106	.9004
4°	.0698	.9976	.0699	14.300	49°	.7547	.6561	1.1504	.8693
5°	.0872	.9962	.0875	11.430	50°	.7660	.6428	1.1918	.8391
6°	.1045	.9945	.1051	9.5144	51°	.7771	.6293	1.2349	.8098
7°	.1219	.9925	.1228	8.1443	52°	.7880	.6157	1.2799	.7813
8°	.1392	.9903	.1405	7.1154	53°	.7986	.6018	1.3270	.7536
9°	.1564	.9877	.1584	6.3138	54°	.8090	.5878	1.3764	.7265
10°	.1736	.9848	.1763	5.6713	55°	.8192	.5736	1.4281	.7002
11°	.1908	.9816	.1944	5.1446	56°	.8290	.5592	1.4826	.6745
12°	.2079	.9781	.2126	4.7046	57°	.8387	.5446	1.5399	.6494
13°	.2250	.9744	.2309	4.3315	58°	.8480	.5299	1.6003	.6249
14°	.2419	.9703	.2493	4.0108	59°	.8572	.5150	1.6643	.6009
15°	.2588	.9659	.2679	3.7321	60°	.8660	.5000	1.7321	.5774
16°	.2756	.9613	.2867	3.4874	61°	.8746	.4848	1.8040	.5543
17°	.2924	.9563	.3057	3.2709	62°	.8829	.4695	1.8807	.5317
18°	.3090	.9511	.3249	3.0777	63°	.8910	.4540	1.9626	.5095
19°	.3256	.9455	.3443	2.9042	64°	.8988	.4384	2.0503	.4877
20°	.3420	.9397	.3640	2.7475	65°	.9063	.4226	2.1445	.4663
21°	.3584	.9336	.3839	2.6051	66°	.9135	.4067	2.2460	.4452
22°	.3746	.9272	.4040	2.4751	67°	.9205	.3907	2.3559	.4245
23°	.3907	.9205	.4245	2.3559	68°	.9272	.3746	2.4751	.4040
24°	.4067	.9135	.4452	2.2460	69°	.9336	.3584	2.6051	.3839
25°	.4226	.9063	.4663	2.1445	70°	.9397	.3420	2.7475	.3640
26°	.4384	.8988	.4877	2.0503	71°	.9455	.3256	2.9042	.3443
27°	.4540	.8910	.5095	1.9626	72°	.9511	.3090	3.0777	.3249
28°	.4695	.8829	.5317	1.8807	73°	.9563	.2924	3.2709	.3057
29°	.4848	.8746	.5543	1.8040	74°	.9613	.2756	3.4874	.2867
30°	.5000	.8660	.5774	1.7321	75°	.9659	.2588	3.7321	.2679
31°	.5150	.8572	.6009	1.6643	76°	.9703	.2419	4.0108	.2493
32°	.5299	.8480	.6249	1.6003	77°	.9744	.2250	4.3315	.2309
33°	.5446	.8387	.6494	1.5399	78°	.9781	.2079	4.7046	.2126
34°	.5592	.8290	.6745	1.4826	79°	.9816	.1908	5.1446	.1944
35°	.5736	.8192	.7002	1.4281	80°	.9848	.1736	5.6713	.1763
36°	.5878	.8090	.7265	1.3764	81°	.9877	.1564	6.3138	.1584
37°	.6018	.7986	.7536	1.3270	82°	.9903	.1392	7.1154	.1405
38°	.6157	.7880	.7813	1.2799	83°	.9925	.1219	8.1443	.1228
39°	.6293	.7771	.8098	1.2349	84°	.9945	.1045	9.5144	.1051
40°	.6428	.7660	.8391	1.1918	85°	.9962	.0872	11.430	.0875
41°	.6561	.7547	.8693	1.1504	86°	.9976	.0698	14.300	.0699
42°	.6691	.7431	.9004	1.1106	87°	.9986	.0523	19.081	.0524
43°	.6820	.7314	.9325	1.0724	88°	.9994	.0349	28.636	.0349
44°	.6947	.7193	.9657	1.0355	89°	.9998	.0175	57.290	.0175
45°	.7071	.7071	1.0000	1.0000	90°	1.0000	.0000	∞	.0000

The process of interpolation depends on the assumption that a small change in the ratio is proportional to the change in the corresponding angle for the degree intervals.

EXERCISES

1. Find sin 26° 30′ and sin 26° 20′.

Solution. From the table sin 26° = .4384 and sin 27° = .4540. Sin 26° 30′ lies halfway between these two numbers or at .4462. Sin 26° 20′ would be one third of the way from .4384 to .4540 or at .4436.

2. Find A if $\sin A = .3365$.

Solution. Reference to the table shows that .3365 lies between .3256 and .3420. Hence A must lie between 19° and 20°. The entire difference between .3420 and .3256 is .0164, which corresponds to a change of one degree or 60 minutes. But .3365 is .0109 greater than .3256. Hence .0109 corresponds to $\frac{109}{164} \times 60$ or 40 minutes of the difference between 19° and 20°. Therefore angle $A = 19° 40′$.

3. Find the value of the function for:

(*a*) sin 27° 20′.
(*b*) sin 43° 50′.
(*c*) sin 69° 10′.
(*d*) tan 18° 40′.
(*e*) tan 82° 20′.
(*f*) tan 64° 48′.
(*g*) sin 71° 24′.
(*h*) tan 18° 18′.
(*i*) sin 24° 24′.

4. Find the angle A if:

(*a*) $\sin A = .4924$.
(*b*) $\sin A = .9304$.
(*c*) $\sin A = .3638$.
(*d*) $\sin A = .7030$.
(*e*) $\tan A = .8355$.
(*f*) $\tan A = .8591$.
(*g*) $\tan A = .7046$.
(*h*) $\tan A = .4417$.
(*i*) $\sin A = .2108$.
(*j*) $\sin A = .9652$.
(*k*) $\tan A = .0787$.
(*l*) $\tan A = 2.1775$.

Note. The sine and the tangent increase as the angle increases, while the cosine and the cotangent decrease as the angle increases. A moment's inspection of the definitions and the triangle of § 246 will verify this extremely important fact, and a glance at the table on page 555 will further confirm it. Hence it is important to observe that while the process of interpolation is similar in the two cases it is not identical.

5. Find cos 18° 40′.

Solution. From the table cos 18° = .9511 and cos 19° = .9455. But 40′ corresponds to $\frac{40}{60}$ of .0056 = .0037. Therefore cos 18° 40′ = .9511 minus .0037 = .9474.

6. Find A if cot A = 1.1237.

Solution. Reference to the table shows that 1.1237 lies between 1.1504 the cot 41° and 1.1106 the cot 42°. The difference between these two cotangents is .0398. The difference between 1.1504 and 1.1237 is .0267 and $\frac{267}{398}$ of 60′ = 40′. Therefore angle A = 41° 40′.

7. Find the value of the function for:

(*a*) cos 32° 40′.
(*b*) cos 23° 20′.
(*c*) cot 27° 40′.
(*d*) cot 35° 10′.
(*e*) cos 41° 50′.
(*f*) cos 54° 18′.
(*g*) cot 67° 20′.
(*h*) cot 38° 50′.
(*i*) cos 25° 10′.

8. Find the angle if:

(*a*) cos A = .8689.
(*b*) cos A = .9717.
(*c*) cos A = .2193.
(*d*) cos A = .8732.
(*e*) cot A = 1.7090.
(*f*) cot A = 2.3945.
(*g*) cot A = 1.
(*h*) cos A = 0.
(*i*) sin A = 0.
(*j*) tan A = 0.
(*k*) tan A = 1.
(*l*) cos A = .8465.
(*m*) cos A = .3886.
(*n*) cos A = .8355.
(*o*) cos A = .8021.
(*p*) cot A = 1.5900.
(*q*) cot A = .9217.
(*r*) cos A = .7790.

Find the missing parts in the following right triangles, using logarithms in the computation if the teacher prefers.

	a	b	c	A	B
9.	24	?	?	18° 40′	?
10.	?	625	?	?	40° 20′
11.	?	?	1200	51° 20′	?
12.	3	4	?	?	?
13.	12	5	?	?	?
14.	28	45	?	?	?
15.	?	55	73	?	?
16.	136	?	305	?	?
17.	1236	?	?	81° 40′	?
18.	4380	?	?	27° 50′	?

248. Logarithms of the trigonometric functions. Since the solution of triangles and other uses of trigonometry also require both multiplication and division, the actual labor involved is much shortened by using a table of the logarithms of the functions instead of the functions themselves. Such a table is given on page 560. Interpolation with such a table is precisely the same as with a table of natural functions.

In using the tables on page 560 it is important to note that the sines and cosines of all angles from zero to ninety degrees are less than one except cosine 0° and sin 90°. Also the tangents of all angles less than 45° and the cotangents of all angles greater than 45° are less than one. Hence the logarithms of these functions will be negative, and minus 10 is *understood* after each in the table.

On the other hand, the tangents of angles greater than 45° and the cotangents of angles less than 45° are greater than one, and hence their logarithms are *positive* and are printed in full.

The foregoing can be reduced to one simple statement:

Minus 10 *is understood after all logarithms in the table except those in the column at the top of which is the abbreviation "COT."*

EXERCISES

Solve the following, using the logarithmic tables on page 560:

1. In the right triangle ABC side $AC = 9.86$ and $\angle A = 34° 20'$. Solve the triangle.

Solution.

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

Hence

$$\log a = \log b + \log \tan A$$

$$= \log 9.86 + \log \tan 34° 20'.$$

$$\begin{array}{rl} \log 9.86 = & .9939 \\ \log \tan 34° 20' = & 9.8344 - 10 \\ \hline \log a = & .8283 \\ a = & 6.734. \end{array}$$

$$\sin A = \frac{a}{c}$$

or

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a - \log \sin A, \text{ etc.}$$

Compute the missing parts in the following right triangles.

	a	b	c	A	B
2.	187	?	?	41° 40′	?
3.	?	64.9	?	19° 20′	?
4.	?	?	.0785	?	35° 20′
5.	154	?	200	?	?
6.	609	481	?	?	?

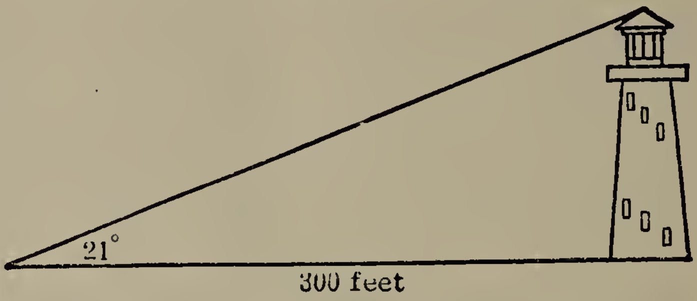
LOGARITHMS OF TRIGONOMETRIC RATIOS

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
0°	– ∝	10.0000	– ∝	∝	**90°**	**11°**	9.2806	9.9919	9.2887	0.7113	**79°**
15′	7.6398	.0000	7.6398	2.3602	45′	15′	.2902	.9916	.2987	.7013	45′
30′	.9408	.0000	.9409	.0591	30′	30′	.2997	.9912	.3085	.6915	30′
45′	8.1169	.0000	8.1170	1.8830	15′	45′	.3089	.9908	.3181	.6819	15′
1°	8.2419	9.9999	8.2419	1.7581	**89°**	**12°**	9.3179	9.9904	9.3275	0.6725	**78°**
15′	.3388	.9999	.3389	.6611	45′	15′	.3267	.9900	.3367	.6633	45′
30′	.4179	.9999	.4181	.5819	30′	30′	.3353	.9896	.3458	.6542	30′
45′	.4848	.9998	.4851	.5149	15′	45′	.3438	.9892	.3546	.6454	15′
2°	8.5428	9.9997	8.5431	1.4569	**88°**	**13°**	9.3521	9.9887	9.3634	0.6366	**77°**
15′	.5939	.9997	.5943	.4057	45′	15′	.3602	.9883	.3719	.6281	45′
30′	.6397	.9996	.6401	.3599	30′	30′	.3682	.9878	.3804	.6196	30′
45′	.6810	.9995	.6815	.3185	15′	45′	.3760	.9874	.3886	.6114	15′
3°	8.7188	9.9994	8.7194	1.2806	**87°**	**14°**	9.3837	9.9869	9.3968	0.6032	**76°**
15′	.7535	.9993	.7542	.2458	45′	15′	.3912	.9864	.4048	.5952	45′
30′	.7857	.9992	.7865	.2135	30′	30′	.3986	.9859	.4127	.5873	30′
45′	.8156	.9991	.8165	.1835	15′	45′	.4059	.9855	.4204	.5796	15′
4°	8.8436	9.9989	8.8446	1.1554	**86°**	**15°**	9.4130	9.9849	9.4281	0.5719	**75°**
15′	.8699	.9988	.8711	.1289	45′	15′	.4200	.9844	.4356	.5644	45′
30′	.8946	.9987	.8960	.1040	30′	30′	.4269	.9839	.4430	.5570	30′
45′	.9181	.9985	.9196	.0804	15′	45′	.4337	.9834	.4503	.5497	15′
5°	8.9403	9.9983	8.9420	1.0580	**85°**	**16°**	9.4403	9.9828	9.4575	0.5425	**74°**
15′	.9614	.9982	.9633	.0367	45′	15′	.4469	.9823	.4646	.5354	45′
30′	.9816	.9980	.9836	.0164	30′	30′	.4533	.9817	.4716	.5284	30′
45′	9.0008	.9978	9.0030	0.9970	15′	45′	.4597	.9812	.4785	.5215	15′
6°	9.0192	9.9976	9.0216	0.9784	**84°**	**17°**	9.4659	9.9806	9.4853	0.5147	**73°**
15′	.0369	.9974	.0395	.9605	45′	15′	.4721	.9800	.4921	.5079	45′
30′	.0539	.9972	.0567	.9433	30′	30′	.4781	.9794	.4987	.5013	30′
45′	.0702	.9970	.0732	.9268	15′	45′	.4841	.9788	.5053	.4947	15′
7°	9.0859	9.9968	9.0891	0.9109	**83°**	**18°**	9.4900	9.9782	9.5118	0.4882	**72°**
15′	.1011	.9961	.1045	.8955	45′	15′	.4958	.9776	.5182	.4818	45′
30′	.1157	.9963	.1194	.8806	30′	30′	.5015	.9770	.5245	.4755	30′
45′	.1299	.9960	.1338	.8662	15′	45′	.5071	.9763	.5308	.4692	15′
8°	9.1436	9.9958	9.1478	0.8522	**82°**	**19°**	9.5126	9.9757	9.5370	0.4630	**71°**
15′	.1568	.9955	.1614	.8387	45′	15′	.5181	.9750	.5431	.4569	45′
30′	.1697	.9952	.1745	.8255	30′	30′	.5235	.9743	.5491	.4509	30′
45′	.1822	.9949	.1873	.8127	15′	45′	.5288	.9737	.5551	.4449	15′
9°	9.1943	9.9946	9.1997	0.8003	**81°**	**20°**	9.5341	9.9730	9.5611	0.4389	**70°**
15′	.2061	.9943	.2118	.7882	45′	15′	.5392	.9723	.5669	.4331	45′
30′	.2176	.9940	.2236	.7764	30′	30′	.5443	.9716	.5727	.4273	30′
45′	.2288	.9937	.2351	.7649	15′	45′	.5494	.9709	.5785	.4215	15′
10°	9.2397	9.9934	9.2463	0.7537	**80°**	**21°**	9.5543	9.9702	9.5842	0.4158	**69°**
15′	.2503	.9930	.2573	.7427	45′	15′	.5592	.9694	.5898	.4102	45′
30′	.2606	.9927	.2680	.7320	30′	30′	.5641	.9687	.5954	.4046	30′
45′	.2707	.9923	.2784	.7216	15′	45′	.5689	.9679	.6009	.3991	15′
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

LOGARITHMS OF TRIGONOMETRIC RATIOS

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
22°	9.5736	9.9672	9.6064	0.3936	**68°**	**34°**	9.7476	9.9186	9.8290	0.1710	**56°**
15′	.5782	.9664	.6118	.3882	45′	15′	.7504	.9173	.8331	.1669	45′
30′	.5828	.9656	.6172	.3828	30′	30′	.7531	.9160	.8371	.1629	30′
45′	.5874	.9648	.6226	.3774	15′	45′	.7559	.9147	.8412	.1588	15′
23°	9.5919	9.9640	9.6279	0.3721	**67°**	**35°**	9.7586	9.9134	9.8452	0.1548	**55°**
15′	.5963	.9632	.6331	.3669	45′	15′	.7613	.9120	.8493	.1507	45′
30′	.6007	.9624	.6383	.3617	30′	30′	.7640	.9107	.8533	.1467	30′
45′	.6050	.9616	.6435	.3565	15′	45′	.7666	.9093	.8573	.1427	15′
24°	9.6093	9.9607	9.6486	0.3514	**66°**	**36°**	9.7692	9.9080	9.8613	0.1387	**54°**
15′	.6135	.9599	.6537	.3463	45′	15′	.7718	.9066	.8652	.1348	45′
30′	.6177	.9590	.6587	.3413	30′	30′	.7744	.9052	.8692	.1308	30′
45′	.6219	.9582	.6637	.3363	15′	45′	.7769	.9038	.8732	.1268	15′
25°	9.6259	9.9573	9.6687	0.3313	**65°**	**37°**	9.7795	9.9023	9.8771	0.1229	**53°**
15′	.6300	.9564	.6736	.3264	45′	15′	.7820	.9009	.8811	.1189	45′
30′	.6340	.9555	.6785	.3215	30′	30′	.7844	.8995	.8850	.1150	30′
45′	.6379	.9546	.6834	.3166	15′	45′	.7869	.8980	.8889	.1111	15′
26°	9.6418	9.9537	9.6882	0.3118	**64°**	**38°**	9.7893	9.8965	9.8928	0.1072	**52°**
15′	.6457	.9527	.6930	.3070	45′	15′	.7918	.8950	.8967	.1033	45′
30′	.6495	.9518	.6977	.3023	30′	30′	.7941	.8935	.9006	.0994	30′
45′	.6533	.9508	.7025	.2975	15′	45′	.7965	.8920	.9045	.0955	15′
27°	9.6570	9.9499	9.7072	0.2928	**63°**	**39°**	9.7989	9.8905	9.9084	0.0916	**51°**
15′	.6607	.9489	.7118	.2882	45′	15′	.8012	.8890	.9122	.0878	45′
30′	.6644	.9479	.7165	.2835	30′	30′	.8035	.8874	.9161	.0839	30′
45′	.6680	.9469	.7211	.2789	15′	45′	.8058	.8858	.9200	.0800	15′
28°	9.6716	9.9459	9.7257	0.2743	**62°**	**40°**	9.8081	9.8843	9.9238	0.0762	**50°**
15′	.6752	.9449	.7302	.2698	45′	15′	.8103	.8827	.9277	.0723	45′
30′	.6787	.9439	.7348	.2652	30′	30′	.8125	.8810	.9315	.0685	30′
45′	.6821	.9429	.7393	.2607	15′	45′	.8148	.8794	.9353	.0647	15′
29°	9.6856	9.9418	9.7438	0.2562	**61°**	**41°**	9.8169	9.8778	9.9392	0.0608	**49°**
15′	.6890	.9408	.7482	.2518	45′	15′	.8191	.8761	.9430	.0570	45′
30′	.6923	.9397	.7526	.2474	30′	30′	.8213	.8745	.9468	.0532	30′
45′	.6957	.9386	.7571	.2429	15′	45′	.8234	.8728	.9506	.0494	15′
30°	9.6990	9.9375	9.7614	0.2386	**60°**	**42°**	9.8255	9.8711	9.9544	0.0456	**48°**
15′	.7022	.9364	.7658	.2342	45′	15′	.8276	.8694	.9582	.0418	45′
30′	.7055	.9353	.7701	.2299	30′	30′	.8297	.8676	.9621	.0379	30′
45′	.7087	.9342	.7745	.2255	15′	45′	.8317	.8659	.9659	.0341	15′
31°	9.7118	9.9331	9.7788	0.2212	**59°**	**43°**	9.8338	9.8641	9.9697	0.0303	**47°**
15′	.7150	.9319	.7831	.2169	45′	15′	.8358	.8624	.9735	.0265	45′
30′	.7181	.9308	.7873	.2127	30′	30′	.8378	.8606	.9772	.0228	30′
45′	.7212	.9296	.7916	.2084	15′	45′	.8398	.8588	.9810	.0190	15′
32°	9.7242	9.9284	9.7958	0.2042	**58°**	**44°**	9.8418	9.8569	9.9848	0.0152	**46°**
15′	.7272	.9272	.8000	.2000	45′	15′	.8437	.8551	.9886	.0114	45′
30′	.7302	.9260	.8042	.1958	30′	30′	.8457	.8532	.9924	.0076	30′
45′	.7332	.9248	.8084	.1916	15′	45′	.8476	.8514	.9962	.0038	15′
33°	9.7361	9.9236	9.8125	0.1875	**57°**	**45°**	9.8495	9.8495	0.0000	0.0000	**45°**
15′	.7390	.9224	.8167	.1833	45′						
30′	.7419	.9211	.8208	.1792	30′						
45′	.7447	.9198	.8249	.1751	15′						
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

7. From a point on a level with the base of a lighthouse a line to its top makes an angle of 21° with the horizontal. The distance of the point from the base of the lighthouse is 300 feet. How high is the lighthouse?

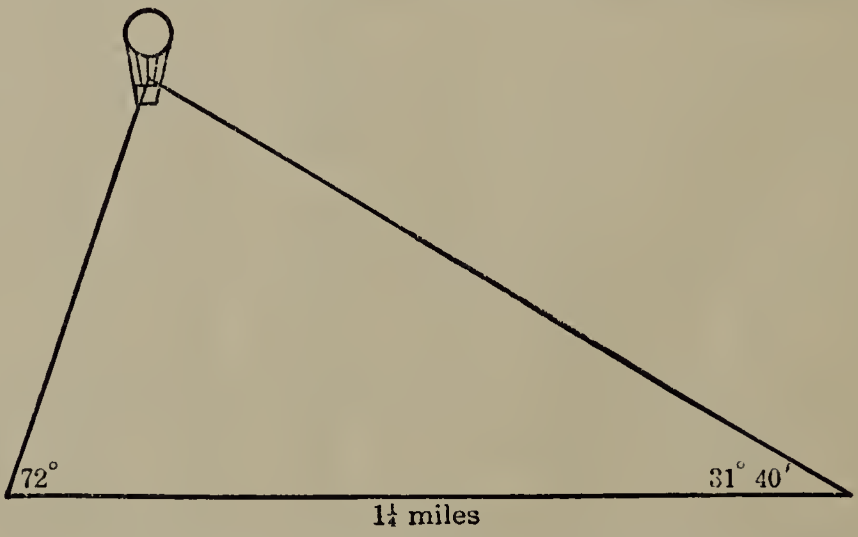


8. A line from a point *A* on the ground to the top of a flag pole makes an angle of 61° with the vertical. The distance to the flag pole from *A* is 320 feet. Find the height of the flag pole above *A*.

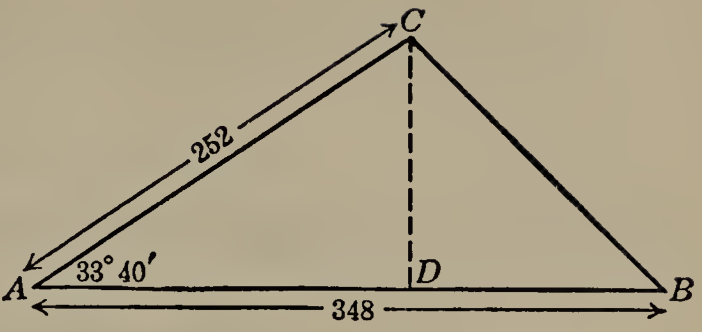
9. A tree casts a shadow 120 feet long when the sun is 42° 20′ above the horizon. Find the height of the tree.

10. The length of a kite string is 1200 yards and the average angle it makes with the horizontal is 39°. How high is the kite?

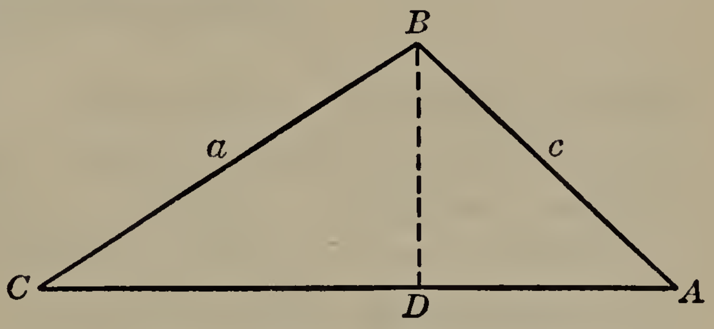
11. Two observers facing each other on a level field noted the elevation of a balloon to be 72° and 31° 40′ respectively. If the distance between the observers is $1\frac{1}{4}$ miles, find the height of the balloon.



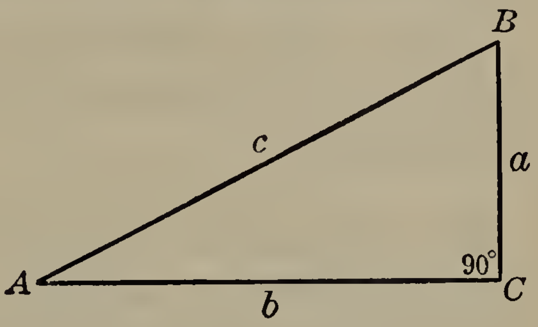
12. Two sides of a triangle are 348 and 252 and their included angle is 33° 40′. Find the altitude of the triangle and its area.



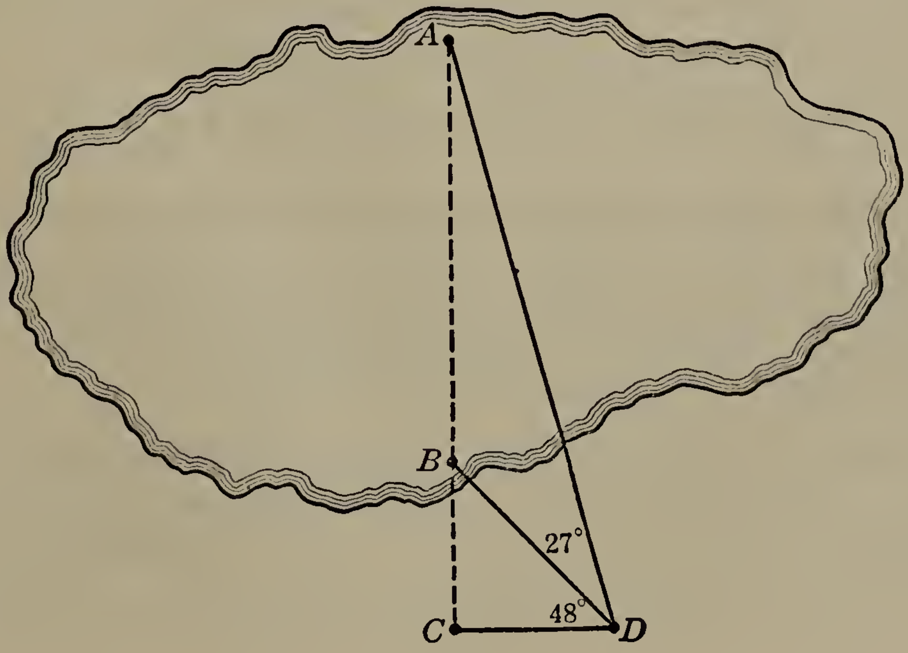
13. Show that in any triangle the altitude $BD = a \sin C$ and the area $= \frac{ab \sin C}{2}$ or $\frac{bc \sin A}{2}$.



14. Show that the area of a right triangle is $\frac{1}{2} bc \sin A$, $\frac{1}{2} b^2 \tan A$, or $\frac{1}{2} a^2 \tan B$.



15. A and B are two channel markers in White Lake. C is a point on the shore, in the line of AB, and CD, 1000 feet long, is perpendicular to AC. The angle BDC is 48° and $CDA = 75°$. Find the distance AB.



CHAPTER XXXVII

PROGRESSIONS

249. A sequence of numbers. In all fields of mathematics we frequently encounter groups of three or more numbers, selected according to some law and arranged in a definite order, whose relations to each other and to other numbers we wish to study.

There is an unlimited variety of such groups, or successions, of numbers. Only two simple types will be considered here.

250. Arithmetic progression. An *arithmetic progression* is a succession of terms in which each term after the first is formed by adding the same number to each preceding number.

Thus, if a denotes the first term and d the common number added, any arithmetic progression is represented by

$$a, a + d, a + 2d, a + 3d, a + 4d, \cdots.$$

This common number d is called the common difference and may be any number, *positive* or *negative.* It may be found for any given arithmetic progression by subtracting any term from the term which follows it.

The numbers 4, 7, 10, 13, $\cdots$ form an arithmetic progression, since any term, after the first, minus the preceding one gives 3. Similarly, 11, 7, 3, -1, -5, $\cdots$ is an arithmetic progression, since any term, after the first, minus the preceding one gives the common difference -4. In like manner $\frac{7}{3}$, 4, $5\frac{2}{3}$, $\cdots$ is an arithmetic progression whose common difference is $1\frac{2}{3}$.

ORAL EXERCISES

State the first four terms of the arithmetic progressions:

1. $a = 3$, $d = 4$.
2. $a = 10$, $d = 7$.
3. $a = 4$, $d = 5$.
4. $a = 15$, $d = 6$.
5. $a = 40$, $d = -1$.
6. $a = 80$, $d = -4$.
7. $a = -5$, $d = -2$.
8. $a = 40\,m$, $d = -2\,m$.
9. $a = m$, $d = 2\,m - 1$.
10. $a = -5$, $d = -2$.

Which of the following are arithmetic progressions:

11. 1, 7, 12.
12. 3, 5, 7.
13. 20, 16, 11.
14. 8, 6, 4.
15. $3\,m$, $4\,m - 2$, $5\,m - 4$.
16. $6\frac{3}{8}$, $9\frac{5}{8}$, $11\frac{7}{8}$.
17. 14, 3, -8.
18. $5\,m$, $4\,m + 2$, $3\,m + 4$.
19. 2, 4, 8.
20. $2\,m - 1$, m, 1, $2 - x$.

21. Give a verbal definition of an arithmetic progression.

22. Give a symbolic definition of an arithmetic progression.

251. The last or *n*th term of an arithmetic progression. In the symbolic definition of an arithmetic progression

$$a,\ a + d,\ a + 2\,d,\ a + 3\,d,\ a + 4\,d,\ \cdots$$

the coefficient of d in the *second* term is 1, in the *third* term it is 2, in the *fourth* term it is 3, etc.; that is, in each term it is one less than the number of the term. Hence, in the nth, or general, term, the coefficient of d is $n - 1$. Hence, if l denotes the nth term, we have

$$l = a + (n - 1)d.$$

EXERCISES

Find the required terms in the following arithmetic progressions:

1. The tenth term of 2, 5, and 8.

Solution. Here $a = 2$, $d = 3$ and $n = 10$.

Hence $l = a + (n - 1)d$

becomes $l = 2 + (10 - 1)3 = 29$.

2. The ninth term of 3, 7, 11, ⋯.

3. The fourteenth term of 8, 6, 4, ⋯.

4. The twentieth term of m, $5m$, $9m$, ⋯.

5. The thirtieth term of -10, $-8\frac{1}{3}$, $-6\frac{2}{3}$, ⋯.

6. The fifteenth term of $\frac{4}{5}$, $\frac{2}{5}$, 0, ⋯.

7. The twentieth term of m, $2m + 1$, $3m + 2$, ⋯.

8. The nth term of 2, 4, 6, ⋯.

9. Determine a general form for any even number.

10. The nth term of 1, 3, 5, ⋯.

11. Determine a general form for any odd number.

12. The nth term of 7, 9, 11, ⋯.

13. $n - 1$st term of the series in Exercises 8, 10, 12.

14. The nth term of the series 8, 10, 12, ⋯.

15. The nth term of $\frac{1}{c}$, $\frac{a-1}{c}$, $\frac{2a-3}{c}$.

16. Find the amount of seventy-five dollars at six per cent simple interest for four years.

17. Find the amount of P dollars for n years at r per cent simple interest.

18. What does the expression in Problem 17 become when $n = 1$? 2? 3? Are these three results in arithmetic progression?

19. A body falls 16 feet the first second, 48 feet the next, 80 feet the next, and so on. How far does it fall during the tenth second? the twentieth? the nth?

20. The digits of a certain three-digit number are in arithmetic progression. The sum of the digits is 15 and the sum of their squares is 93. Find the number.

21. If an arithmetic progression has an odd number of terms, show that the middle term is half the sum of the first and last terms.

22. In an arithmetic progression find the 9th term from the beginning; the 9th term from the end.

23. The pairs of numbers $(1, \frac{3}{4})$, $(2, \frac{3}{2})$, $(3, \frac{9}{4})$, $(4, 3)$, $(5, \frac{15}{4})$, $(6, -\frac{9}{2})$ are respectively the x and y of coördinates of six points. Show that the distances of the points from the origin are in arithmetic progression.

24. Find the rth term from the beginning of an arithmetic progression and the rth term from the end. Show that the arithmetic average of these two terms is the middle term of the progression if it has an odd number of terms.

25. The velocity of a body falling from rest increases uniformly 32 feet per second each second. Find the velocity at the end of the 10th second. Find the average velocity during the first second, during the third second, during the nth second.

252. Arithmetic means. The arithmetic mean between two numbers is a number which forms with the two given ones as the first and last terms an arithmetical progression.

One important special case of arithmetic means is the arithmetic *mean between two numbers h and k.* This is $\frac{h+k}{2}$ or merely the average of the two numbers.

Proof. There would be three terms in the progression, h being the first and k the last.

Then $\quad l = a + (n - 1)d$

becomes $\quad k = h + (3 - 1)d.$

Hence $\quad d = \frac{k-h}{2}.$

Therefore the series is $h, h + \left(\frac{k-h}{2}\right), k$

or $\quad h, \frac{h+k}{2}, k.$

EXERCISES

1. Insert five arithmetic means between 8 and 92.

Solution. There will be seven terms in all.

Therefore $\quad l = a + (n - 1)d$

becomes $\quad 92 = 8 + (7 - 1)d = 8 + 6d.$

Hence $\quad d = 14.$

Therefore the series is 8, 22, 36, 50, 64, 78, 92.

2. Insert 4 arithmetic means between 100 and 200.

3. Insert 10 arithmetic means between 11 and 99.

4. Insert 6 arithmetic means between 70 and 800.

Find the arithmetic mean between:

5. 18, 60.

6. 30, 400.

7. a, c.

8. $x, x - 2y$.

9. $x, -x$.

10. $a + b$ and $a - b$.

11. What is the arithmetic mean of the odd numbers from 23 to 37 inclusive?

12. An automobile starting from rest increased its speed in 18 seconds to 66 feet per second. What was its final rate in miles per hour? What was its average rate in miles per hour during the time its speed was increasing?

253. The sum of a series. The indicated sum of several terms of an arithmetic progression is called an arithmetic series. The formula for the sum of n terms of an arithmetic series is derived as follows:

$$S_n = a + (a + d) + (a + 2d) + \cdots [a + (n - 3)d] + [a + (n - 2)d] + [a + (n - 1)d] \qquad (1)$$

Writing (1) again and reversing the order of the terms in the right member gives:

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \cdots (a + 2d) + (a + d) + a \qquad (2)$$

Adding (1) and (2) gives

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \cdots [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] \qquad (3)$$

Since there are n terms in the right members of (1) and (2), there are n brackets in (3); therefore

$2S_n = n[2a + (n - 1)d]$ or

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

EXERCISES

Find the sum in the following arithmetic progressions:

1. The first 18 terms of 3, 5, 7.

Solution. Here $a = 3$, $d = 2$, and $n = 18$.

Hence $$s = \frac{n}{2}[2a + (n-1)d]$$

becomes $$s = \tfrac{18}{2}[2 \cdot 3 + (18-1)2] = 9(6+34) = 360.$$

2. The first 50 terms of $1 + 2 + 3 + \cdots$.

3. The first 8 terms of $60 + 57 + 54 + \cdots$.

4. The first 10 terms of $1 + 1\frac{1}{2} + 2 + \cdots$.

5. All the even numbers less than 101.

6. All the odd numbers between 100 and 200.

7. The sum of the first ten thousand integers.

8. The sum of the 20 terms of $2 + 2.2 + 2.4 + \cdots$.

9. Find the sum of the integers between 200 and 600 which are exact multiples of 7.

10. Find the sum of the numbers less than 100 which are exact multiples of 11.

11. The sum of 15 terms of $(3 + \sqrt{2}) + 3 + (3 - \sqrt{2}) + \cdots$.

12. The sum of 10 terms of $\frac{1}{c} + \frac{c-1}{c} + \frac{2c-3}{c} + \cdots$.

13. The sum of n terms of $(a - c) + a + (a + c) + \cdots$.

14. The sum of n terms of $3c + c - c \cdots$.

15. The sum of n terms of $4\frac{1}{2} + 3\frac{1}{2} + 2\frac{1}{2} + \cdots$.

16. The sum of n terms of $1 + 3 + 5 + \cdots$.

17. The sum of n terms of $2 + 4 + 6 + \cdots$.

18. Find the sum of the first n integers which are exact multiples of 13.

19. How many terms of $1 + 5 + 9 + \cdots$ will amount to 630?

HINT. We have here S, a, and d in $S = \frac{n}{2}[2a + (n-1)d]$, to solve the equation for n.

20. How many terms of $15 + 12\frac{1}{2} + 10 + \cdots$ will amount to 15?

21. How many terms of $-12 - 10\frac{1}{2} - 9 \cdots$ will amount to -54?

22. How many terms of $16 + 15\frac{2}{3} + 15\frac{1}{3} + \cdots$ will amount to 216?

23. The first term of an arithmetic progression is 7 and the 6th term is 12. Find the sum of the first 100 terms.

24. Assuming that a heavy iron ball is not retarded by the air, determine the number of seconds it would take to fall if it is dropped from a point on the Eiffel Tower 980 feet above the ground. ($S = 16t^2$.)

25. A ball shot vertically upward returned to the point it started from 18 seconds later. How high did it rise?

26. By Exercise 25, page 567, it is seen that a falling body obeys the law of an arithmetic progression. Show from the data of that exercise that the general formula

$$S = \frac{n}{2}[2a + (n-1)d]$$

becomes the special one, $S = \frac{gt^2}{2}$ used in physics for such problems.

27. A passenger in an aëroplane drops a rock and observes that it strikes the ground 11 seconds later. How high was the plane at the time?

28. A man deposits in a bank 18 dollars on the first of each month from January 1, 1924, to December 1, 1925. What is the amount of his deposit at that time?

29. The bank in Exercise 28 pays three per cent interest, compounded monthly, on deposits. What will the interest amount to by January 1, 1925?

30. A man buys a piano for $900 by paying $200 cash and the remainder in monthly instalments of $100 and interest at six per cent on the unpaid balance. What was his total payment?

HINT. The monthly balances are respectively $700, $600, $500, etc. The corresponding interest payments are $3.50, $3.00, $2.50, etc.

31. A man bought a house for $11,500, paying $1500 cash and giving a first mortgage for $6000 and a second mortgage for $4000, each bearing 6% interest. Payments of $200 plus the accrued interest were made every four months on the second mortgage. Find the sum of all interest payments on the second mortgage.

32. Two men 2010 feet apart start at the same time and move toward each other. One moves 8 feet the first second, 13 feet the second, and 18 feet the third, etc. The other moves 7 feet the first second, 11 the second, 15 the third, and so on. In how many seconds will they meet?

33. A pyramid of balls all of the same size stands on an equilateral triangle, twelve balls on a side. How many balls are there in each layer? in the whole pyramid?

34. A and B set out from the same place at the same time and travel in the same direction. A travels 20 miles per hour. B goes 32 miles the first hour, 28 the second, 24 the third, and so on. When will they be together again?

35. A man leaves a certain place and travels 16 miles the first hour, 18 the second, 20 the third, and so on. One hour and twenty minutes later a second man leaves the same place and travels the same road at a constant rate of 30 miles per hour. When are the two together?

36. The successive intervals between a number of stones placed in a straight line on the ground is 1 rod, 3 rods, 5 rods, etc. A boy collects the stones one by one and places them in a box located at the first stone. How far will he travel in handling the 7th stone?

37. In Exercise 36 determine how far the boy will travel for the nth stone; the $n + 1$st stone; the $n - 1$st stone.

38. From the results of Exercise 37 determine the series which represents the total distance the boy must travel in collecting the first n stones. Can this series be summed by any formula developed thus far?

NOTE. In the earliest mathematical work known a problem is found which involves the idea of an arithmetic progression. In the papyrus of the Egyptian priest Ahmes, who lived nearly two thousand years before Christ, we read in essence, "Divide 40 loaves among 5 persons so that the number of loaves that they receive shall form an arithmetic progression, and so that the two who receive the least bread shall together have one seventh as much as the others." From that time to this, the subject has been a favorite one with mathematical writers, and has been extended so widely that it would require many volumes to record all of the discoveries regarding the various kinds of series.

254. Geometric progression. A *geometric progression* is a succession of terms in which each term after the first is obtained from the preceding one by multiplying it by the same number.

If a denotes the first term and r the common multiplier, the progression is represented by $a, ar, ar^2, ar^3, ar^4, \cdots$.

The common multiplier is called the *ratio*. It is evident from the definition that the ratio is obtained by dividing any term except the first by the preceding one.

The numbers 2, 6, 18, 54, $\cdots$ form a geometrical progression since any term after the first divided by the preceding one gives the same number 3. Similarly, the numbers $4, -4\sqrt{5}, 20, -20\sqrt{5}, \cdots$ since from the second term on each term divided by the preceding one gives the common ratio $-\sqrt{5}$.

ORAL EXERCISES

Determine which of the following are geometric progressions and the ratio:

1. $1, 4, 8, 16, \cdots$.

2. $3, 9, 27, \cdots$.

3. $2, 4, 6, 8, \cdots$.

4. $2, 1, \frac{1}{2}, \cdots$.

5. $6, 2, \frac{1}{3}, \cdots$.

6. $3, \frac{1}{3}\sqrt{3}, \frac{1}{9}\sqrt{3}, \cdots$.

7. $\frac{1}{\sqrt{8^3}}, \frac{\sqrt{2}}{2}, 2, \cdots$.

8. $5a^2, 10a^2x, 20a^2x^2, \cdots$.

9. $10, 6, \frac{18}{5}, \cdots$.

10. $ax^{\frac{1}{2}}, 4a^{\frac{1}{2}}, 16x^{\frac{3}{2}}, \cdots$.

11. Find the condition under which a, b, and c form a geometric progression.

State in order the first four terms of a geometric progression in which the ratio r and the first term a are given:

12. $a = 5$, and $r = 2$.

13. $a = 7$, and $r = 3$.

14. $a = 10$, and $r = -2$.

15. $a = 6$, and $r = \frac{1}{3}$.

16. $a = \frac{4}{5}$, and $r = -\frac{1}{2}$.

17. $a = 7$, and $r = \sqrt{2}$.

255. The *n*th term of a geometric progression. In the symbolic definition of a geometrical progression

$$a, ar, ar^2, ar^3, ar^4, \cdots$$

the exponent of r in the second term is 1, in the third term 2, in the fourth term 3, etc.; that is, in each term it is one less than the number of the term. Therefore the nth or general term is

$$t_n = ar^{n-1}$$

EXERCISES

1. Find the sixth term of 3, 6, 12, $\cdots$.

Solution. Here $a = 3$, $r = 2$, and $n = 6$.

Therefore $t_n = ar^{n-1}$

becomes $t_n = 3 \cdot 2^5 = 3 \cdot 32 = 96.$

2. Find the seventh term of 2, 4, 8, $\cdots$.

3. Find the eighth term of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\cdots$.

4. Find the tenth term of 5, -10, 20, $\cdots$.

5. Find the t_9 of 12, -4, $\frac{4}{3}$, $\cdots$.

6. Find the t_{10} of $24a^2$, $18a^4$, $\dfrac{27a^6}{2}$, $\cdots$.

7. Find t_8 of $9\sqrt{3}$, 9, $3\sqrt{3}$, $\cdots$.

8. Find t_7 of $\frac{1}{4}\sqrt{2}$, $\frac{1}{2}$, $\frac{1}{2}\sqrt{2}$, $\cdots$.

9. The number .23232323 is the sum of four terms in geometric progression. What are these terms?

10. The amount, A, of P dollars at r per cent interest compounded annually for n years is expressed by the formula $A = P(1 + r)^n$. Derive this formula.

11. Find the amount of \$500 at four per cent interest compounded annually for 4 years.

12. The first two terms of a geometrical progression are a and b. Find the third term.

13. Find the amount of 100 dollars at $5\frac{1}{2}\%$ compounded annually for 3 years.

14. What is the $(n-1)$st term of a geometric progression? the $(n-2)$d? the $(n-3)$d? the $(n+1)$st?

15. The sum of the digits of a three-digit number is 14 and the digits are in geometric progression. If 594 is added to the number, the result is expressed by the digits in reverse order. Find the number.

16. Show that the distances of the following points from the origin are in geometric progression: $(1, 3)$, $(\sqrt{2}, 3\sqrt{2})$, $(2, 6)$, $(2\sqrt{2}, 6\sqrt{2})$.

17. A house worth $8000 depreciates 10% in value each year. What will it be worth at the end of 7 years?

18. An automobile selling at $800 depreciates in use yearly 30% of its cost. What is it worth at the end of 3 years?

19. Show that the product of any two terms of a geometric progression equidistant from a given term is always the same.

20. Show that if each term of a geometric progression is subtracted from the following one, the successive remainders form a geometric progression.

21. One hundred dollars is deposited annually for six years in a bank which pays four per cent compound interest. What sum is due the depositor at the end of six years if he has withdrawn no money from the bank during that time?

22. A father deposits a certain sum in a bank at the beginning of the school year each of the four years his son is in the high school. If this amounts to sixteen hundred dollars when the boy enters college after his high school course, find the yearly deposit if interest is compounded annually at four per cent.

23. A good ear of Dent corn has about 400 grains. Five per cent of the grains on such an ear are unfit to plant. If the others are planted and each produces an ear like the first, how many bushels of corn will there be in the fifth year's crop if each ear weighs 15 ounces and there are 70 pounds of corn in a bushel?

256. Geometric means. The geometric means between any two numbers are numbers which form, with the given ones as the first and last terms, a geometric progression.

One important special case is the *geometric mean* between two numbers h and k. This is $\pm\sqrt{h \cdot k}$.

Proof. There would be three terms in the progression, h being the first and k the last.

Then $$t_n = ar^{n-1}$$

becomes $$k = hr^2.$$

Hence $$r = \pm\sqrt{\frac{k}{h}}.$$

$$\therefore \text{ the second term is } h\left(\pm\sqrt{\frac{k}{h}}\right) = \pm\sqrt{\frac{k \cdot h^2}{h}} = \pm\sqrt{kh}.$$

Hence the series is $h, \pm\sqrt{hk}, k$.

EXERCISES

1. Insert three geometric means between 3 and 48.

Solution. The number of terms is five.

Therefore $$48 = 3 \cdot r^4.$$

Hence $$r^4 = 16 \text{ and } r = \pm 2.$$

$\therefore$ the series is 3, ± 6, 12, ± 24, 48.

2. Insert four geometric means between 1 and 243.

3. Insert two geometric means between 19 and 152.

4. Insert three geometric means between -7 and -112.

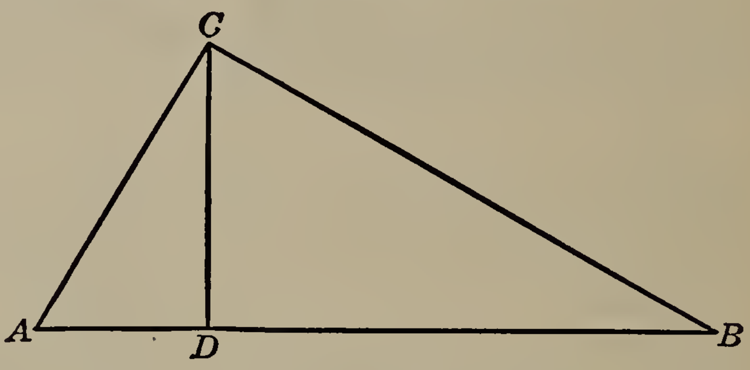
5. Insert one geometric mean between b and c.

6. Insert two geometric means between b and c.

7. The second and third terms of a geometric progression are a and b. Find the first term.

8. The arithmetic mean of two numbers is 26 and their geometric mean is 24. Find the numbers.

9. In the right triangle CD is perpendicular to the hypotenuse AB. It is proved in geometry that CD is a geometric mean between AD and BD.

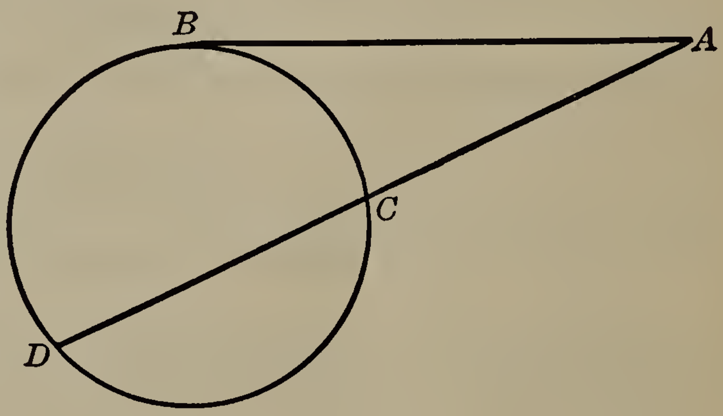


(a) If $AD = 9$ and $BD = 16$, find CD.

(b) If $CD = 6$ and $BC = 13$, find AD and BD.

(c) If $DC = 20$ and $AD = 15$, find BD, AC, and BC.

10. In the adjacent figure AB touches and AD intersects the circle. It is proved in geometry that AB is a geometric mean between AC and AD.



(a) If $AB = 10$ and AD is 90, find AC.

(b) If $DC = 7$ and $AB = 12$, find AC and AD.

257. Geometric series. The indicated sum of n terms of a geometric progression is called a *geometric series.*

In symbols this sum, S_n, is indicated by

$$S_n = a + ar + ar^2 + ar^3 + \cdots ar^{n-1}.$$

In practice the simplest form of this indicated sum is needed. The value of S_n in the simplest form is derived as follows:

$$S_n = a + ar + ar^2 + ar^3 + \cdots ar^{n-3} + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying both members of (1) by (r) gives

$$S_nr = ar + ar^2 + ar^3 + ar^4 \cdots ar^{n-2} + ar^{n-1} + ar^n. \quad (2)$$

Subtracting (2) from (1), all terms on the right vanish except a and ar^n, giving

$$S_n - S_nr = a - ar^n. \quad (3)$$

$$S_n(1 - r) = a - ar^n. \quad (4)$$

Solving (4),

$$S_n = \frac{a - ar^n}{1 - r}.$$

EXERCISES

1. Find the sum of the first eight terms of $2 + 6 + 18 + \cdots$.

Solution. Here $a = 2$, $r = 3$, and $n = 8$.

Substituting in $S_n = \dfrac{a - ar^n}{1 - r}$ gives

$$S_8 = \frac{2 - 2 \cdot 3^8}{1 - 3} = \frac{2 - 2 \cdot 3^8}{-2}$$
$$= -1 + 3^8 = -1 + 6561 = 6560.$$

2. Find the sum of nine terms of $3 + 6 + 12 + \cdots$.

3. Find the sum of seven terms of $5 - 10 + 20 - \cdots$.

4. Find the sum of eight terms of $80 + 40 + 20 + \cdots$.

5. Find the sum of ten terms of $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

6. Find S_7 for $2\sqrt{2} - 4 + 4\sqrt{2} + \cdots$.

7. Find S_9 for $3 + \frac{3}{2} + \frac{3}{4} + \cdots$.

8. Find S_{10} for $\frac{3}{4} + 1 + \frac{4}{3} + \cdots$.

9. Find S_9 for $x^2 + x^4 + x^6 + \cdots$.

10. Find S_8 for $m + 2\,m^3 + 4\,m^5 + \cdots$.

11. Find the sum of n terms of $5 + 10 + 20 + \cdots$.

12. Find the sum of $n - 1$ terms of $c + c^4 + c^7 + \cdots$.

13. Find the sum of seven terms of $\frac{a}{c} + \frac{c}{a} + \frac{c^3}{a^3} + \cdots$.

14. Find the sum of $n - 3$ terms of $\frac{a}{c} + 1 + \frac{c}{a} + \cdots$.

15. The sum of the first three terms of a geometric progression is 65 and the difference of the first and third is 40. Find the terms.

16. The sum of the first four terms of a geometric progression is 30 and the sum of the next four is 480. Find the series.

17. The sum of three numbers which are in arithmetic progression is 21. If 1, 5, and 25 be added to them respectively, the results are in geometric progression. Find the numbers.

18. A rubber ball falls to the floor from a height of 72 inches. At each rebound it rises thirty per cent of its previous height. How far has it moved when it strikes the floor the tenth time?

19. A vessel containing alcohol was emptied of one fourth of its contents and then filled up with water. This was done five times. What portion of the original contents was then in the vessel?

20. At each stroke an air pump withdraws 20 cubic inches of air from a vessel containing 200 cubic inches. After every stroke the air remaining in the vessel expands and completely fills it. What fraction of the original quantity of air remains in the vessel at the end of the eighth stroke?

258. Infinite geometric series. If the number of terms of a geometric series is unlimited, it is called an infinite geometric series.

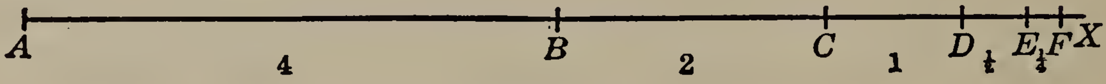
In the progression 3, 6, 12, ⋯ the ratio is positive and greater than 1 and each term is greater than the preceding one. Such a progression is an *increasing* one. It is obvious that the sum of an unlimited number of terms in an increasing geometric progression is unlimited. If $r > 1$ (read " r is greater than 1 "), the sum can be made as large as we please by taking a sufficient number of terms.

If the geometric progression is a decreasing one, however, the result is very different. In the decreasing progression 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ⋯ the ratio is positive and less than 1, and each term is less than the preceding one. Here the sum of the first three terms is 7, the first four $7\frac{1}{2}$, the first five $7\frac{3}{4}$, the first six $7\frac{7}{8}$, etc. Obviously, the more terms taken the greater is the sum. But no matter how large the number of terms taken their sum is less than 8. This number 8 is called the *limit* of the series.

$$S_n = 4 + 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} \cdots.$$

This series and its limit can be illustrated by the motion of a point along a line in the diagram on the following page.

Suppose line AX is eight inches long and a point starts from A and moves in one minute to point B which is $\frac{1}{2}$ the distance to X. In the next minute it moves to C, $\frac{1}{2}$ the remaining distance. In the third minute it moves to D, $\frac{1}{2}$ the remaining distance, etc. But the point will never reach X nor will the total distance it moves ever quite equal eight inches.



The formula $S_n = \dfrac{a - ar^n}{1 - r}$

becomes $$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}. \tag{1}$$

For the preceding series we have $a = 4$, $r = \frac{1}{2}$, and $n = n$.

Substituting, $$S_n = \frac{4}{1-\frac{1}{2}} - \frac{4(\frac{1}{2})^n}{1-\frac{1}{2}}. \tag{2}$$

$$S_n = \frac{4}{\frac{1}{2}} - \frac{4(\frac{1}{2})^n}{\frac{1}{2}} = 8 - 4(\tfrac{1}{2})^{n-1}. \tag{3}$$

Now $(\frac{1}{2})^2 = \frac{1}{4}$, $(\frac{1}{2})^3 = \frac{1}{8}$, $(\frac{1}{2})^4 = \frac{1}{16}$, etc. Hence $(\frac{1}{2})^n$ becomes very small if n is great. Consequently $4(\frac{1}{2})^{n-1}$ approaches *zero* when n increases *without limit*. Therefore the right member of (3) approaches 8 if n increases without limit. That is, the limit of S_n is $\dfrac{4}{\frac{1}{2}}$ or 8. In general, then, when we speak of the *sum* of an infinite geometric series we mean the *limit* which the sum approaches as n increases indefinitely.

Moreover, the formula (1) above becomes for a decreasing series

$$S_\infty = \frac{a}{1-r}.$$

EXERCISES

Find the sum of the following infinite geometric series:

1. $3 + 1 + \frac{1}{3} + \cdots$.

Solution. $$s = \frac{a}{1-r}.$$

Substituting, $$s = \frac{3}{1-\frac{1}{3}} = 4\tfrac{1}{2}.$$

2. $8 + 4 + 2 + 1 + \cdots$.

3. $9 + 3 + 1 + \cdots$.

4. $12 + 3 + \frac{3}{4} + \cdots$.

5. $25 + 5 + 1 + \cdots$.

6. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

7. $2 + \sqrt{2} + 1 + \cdots$.

8. $3 - \sqrt{3} + 1 + \cdots$.

9. $1 + x + x^2 + \cdots$ where $x < 1$.

10. $6 + \sqrt{6} + 1 + \cdots$.

11. $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$ where $x > 1$.

12. $(m+n) + 1 + \frac{1}{m+n} + \cdots$.

13. $.323232 \cdots$.

Hint. This equals

$$\frac{32}{100} + \frac{32}{10,000} + \frac{32}{1,000,000} + \cdots$$

14. $.777 \cdots$.

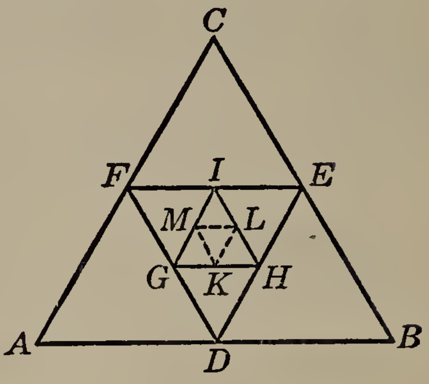
15. $.6464 \cdots$.

16. $.020202 \cdots$.

17. $.714285714285 \cdots$.

18. $3.4242 \cdots$.

19. ABC is an equilateral triangle with side AB equal to 4. The triangle DEF is formed by joining the midpoints of the adjacent sides of ABC. Triangle IGH is formed similarly, and so on. Determine the series formed by the areas of the successive triangles. Find the sum of the areas of all the triangles drawn as there supposed.



20. A flywheel whose circumference is 12 feet makes 2400 revolutions per minute. If it slows down so that each second it makes ninety-eight per cent as many revolutions as it did the preceding second, how far will a point on its rim have traveled by the time it has stopped?

NOTE. In the study of geometric progressions we have seen that the sum of the infinite series $1 + x + x^2 + x^3 + \cdots$ is a definite number when x has any value less than 1. But it has no finite value when x is equal to or greater than 1; that is, we have an expression which we cannot use arithmetically unless x has a properly chosen value. If we were studying a problem which involved such a series, it would be a matter of the most vital importance to know whether the values of x under discussion were such as to make the series meaningless.

This question of distinguishing between expressions which converge (that is, the sum of whose terms approaches a limit) and those which do not has an interesting history. Newton and his followers in the seventeenth century dealt with infinite series and always assumed that they converged, as, in fact, most of them did. But as more complicated series came into use it became more difficult to tell from inspection whether the series converged or not for a given value of the variable.

It was not until the beginning of the nineteenth century that Gauss, Abel, and Cauchy, in Germany, Norway, and France respectively, began to teach this subject effectively and to devise far-reaching tests to determine the values of x for which certain series converge to a finite limit. It is said that on hearing a discussion by Cauchy in regard to series which do not always converge, the astronomer Laplace became greatly alarmed lest he had made use of some such series in his great work, "Celestial Mechanics." He hurried home and denied himself to all distractions until he had examined every series in his book. To his intense satisfaction they all converged.

MISCELLANEOUS EXERCISES

1. Define an arithmetic progression:

(*a*) in symbols, (*b*) in words.

2. Derive the formula for the nth term of an arithmetic progression.

3. Derive the formula for the sum of n terms of an arithmetic progression.

4. What is the arithmetic mean between two numbers?

5. Define a geometric progression:

(*a*) in symbols, (*b*) in words.

6. Derive the formula for the nth term of a geometric progression.

7. Derive the formula for the sum of n terms of a geometric progression.

8. Show that this last formula is equivalent to

$$S_n = \frac{a - rl}{1 - r}.$$

9. Show that for a converging geometric progression

$$S_n = \frac{a}{1 - r}.$$

10. In the equation

$$S(1 + r)^4 - p(1 + r)^3 - p(1 + r)^2 - p(1 + r) - p = 0$$

show that $$p = \frac{Sr(1 + r)^4}{(1 + r)^4 - 1}.$$

11. A loan of S dollars is to be repaid in four equal annual payments of p dollars each. Find p if money is worth r %.

Solution. The sum due at beginning of second year is

$$S(1+r)-p. \tag{1}$$

The sum due at beginning of third year is

$$[S(1+r)-p](1+r)-p. \tag{2}$$

The sum due at beginning of fourth year is

$$\{[S(1+r)-p](1+r)-p\}(1+r)-p. \tag{3}$$

The sum due at beginning of fifth year is

$$[\{[S(1+r)-p](1+r)-p\}(1+r)-p](1+r)-p. \tag{4}$$

By the conditions of the problem, (4) = 0, for all the debt has then been paid. Setting (4) equal to zero and simplifying,

$$S(1+r)^4-p(1+r)^3-p(1+r)^2-p(1+r)-p=0. \tag{5}$$

Solving (5) for p,

$$p=\frac{S(1+r)^4}{(1+r)^3+(1+r)^2+(1+r)+1}. \tag{6}$$

But the denominator in (6) is a geometrical series whose sum by the formula $S_n=\frac{a-ar^n}{1-r}$ is $\frac{(1+r)^4-1}{r}$.

Substituting this last value for the denominator of (6),

$$p=\frac{Sr(1+r)^4}{(1+r)^4-1}. \tag{7}$$

In the general case, if we have n annual payments, the exponents 4 in (7) would be replaced by n, and then

$$p=\frac{Sr(1+r)^n}{(1+r)^n-1}.$$

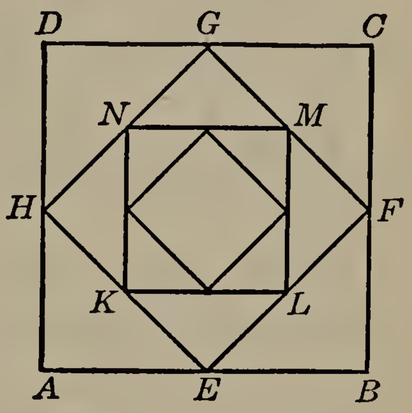
12. A loan of \$600 is to be repaid in four equal annual payments, with interest at 6%. Find the amount of each payment.

13. A loan of \$1000 is to be repaid in four equal annual payments with interest at 5%. Find the payment required.

14. The equipment of a certain factory cost \$25,000. The estimated depreciation in its value is 8% yearly. On that basis compute its value at the end of 5 years.

15. One third the contents of a vessel containing acid was taken out and replaced by water. This was done six times. What portion of the acid remained in the vessel?

16. Each square in the adjacent figure except the first is formed by joining the mid-points of the square next larger. If $AB = 8$, show that the perimeters of the squares form a decreasing geometrical progression. Find the sum of the perimeters of all the squares which may be so drawn.



CHAPTER XXXVIII

RATIO, PROPORTION, AND VARIATION

259. Ratio. The ratio of one number a to a second number b is the quotient obtained by dividing the first by the second, or $\frac{a}{b}$. The ratio of a to b is also written $a : b$.

It follows from the above that all ratios of two numbers are fractions and all fractions may be regarded as ratios.

Thus $\frac{3}{4}$, $\frac{a}{2c}$, $\frac{m+3}{m-n}$, and $\frac{\sqrt{3}}{\sqrt{5}}$ are ratios as well as fractions.

Since ratios like the above are fractions, operations which may be performed on fractions may be performed on these ratios. Hence the value of a ratio is not changed by multiplying or dividing both numerator (antecedent) and denominator (consequent) by the same number.

EXERCISES

Simplify the following ratios by considering them as fractions and reducing the fractions to lowest terms:

1. 15 inches : 1 rod.

2. 100 inches : $\frac{3}{10}$ mile.

3. 2 pints : 7 gallons.

4. 1 mile : 1 kilometer (1 meter = 39.37 inches).

5. .02 ton : 1 kilogram (1 kilogram = 2.205 lbs.).

6. A lot 100 × 135 feet : 1 acre.

7. Which is the greater ratio 17 : 25 or 23 : 33?

8. In an alloy of 52 ounces of zinc and copper there are 12 ounces of zinc. Find the ratio of zinc to copper.

9. Divide $1800 into two parts which are in the ratio of 5 : 7.

10. Separate 360 into three parts in the ratio of 3 : 5 : 7.

11. Show that $\frac{x}{x+3} < \frac{x+4}{x+7}$ if x is positive.

HINT. Reduce the fractions to respectively equivalent fractions having a common denominator and then compare the numerators.

12. Show $\frac{x}{x+5} < \frac{x+3}{x+8}$ if x is positive.

13. Show $\frac{x}{x+3} < \frac{x+4}{x+5}$ if x is positive.

14. Arrange the ratios 3 : 4; 7 : 8; 13 : 16 in decreasing order of magnitude.

15. Simplify $\left(1 - \frac{9}{x^4}\right) : \left(1 + \frac{3}{x^2}\right)$.

16. $\frac{2a^2 - 5a + 3}{a^2 - 9} : \frac{a^2 - 2a + 1}{a^2 - 5a + 6}$.

17. $\frac{a(x^2 - 3a) - ax(2x - 3)}{(x^2 - 3x)^2} : \frac{ax}{x^2 - 3x}$.

18. Which is the greater ratio if a is positive:

$$\frac{7 + 2a}{7 + 3a} \text{ or } \frac{7 + 4a}{7 + 5a}?$$

19. If a positive number is added to or subtracted from both terms of a proper fraction, what change is produced in the value of the fraction?

20. Separate 660 into three parts such that the ratio of the first to the second is 5 : 6 and of the second to the third is 9 : 11.

260. Proportion. A *proportion* is a statement of equality between two ratios. Four numbers, a, b, c, and d, are in proportion if the ratio of the first pair equals the ratio of the second pair.

This proportion is written

$$a:b = c:d \text{ or } \frac{a}{b} = \frac{c}{d}.$$

Though both forms are equations, the second is the more familiar one and for this reason is preferable.

NOTE. By the early mathematicians ratios were not treated as if they were numbers, and the equality of two ratios which we know as a proportion was not denoted by the same symbol as other kinds of equality. The usual sign of equality for ratios was : :, a notation which was introduced by the Englishman Oughtred in 1631 and was brought into common use by John Wallis about 1686. The sign = was used in this connection by Leibnitz (1646–1716) in Germany, and by the Continental writers generally, while the English clung to Oughtred's notation.

In the proportion $\frac{a}{b} = \frac{c}{d}$ the first and fourth terms (a, d) are called the *extremes*, and the second and third terms (b, c) are called the *means*.

261. Mean proportional. A *mean proportional* between two numbers a and b is the number m if $\frac{a}{m} = \frac{m}{b}$. It follows that $m^2 = ab$, or $m = \pm\sqrt{ab}$.

If $\frac{4}{m} = \frac{m}{9}$, $m^2 = 36$ and $m = \pm 6$.

262. Third proportional. A *third proportional* to two numbers a and b is the number t if $\frac{a}{b} = \frac{b}{t}$.

A third proportional to 3 and 5 is t if $\frac{3}{5} = \frac{5}{t}$. Whence $t = \frac{25}{3}$.

Gottfried Wilhelm Leibnitz

263. Fourth proportional. A *fourth proportional* to three numbers a, b, and c is the number f if $\frac{a}{b} = \frac{c}{f}$.

A fourth proportional to 6, 8, and 12 is f, if $\frac{6}{8} = \frac{12}{f}$. Whence $f = 16$.

EXERCISES

1. Find a mean proportional between 16 and 25.

2. Find a third proportional to 16 and 25.

3. Find a fourth proportional to 8, 20, and 32.

4. Find the fourth proportional to a^3, ab, and $5\,a^2b$.

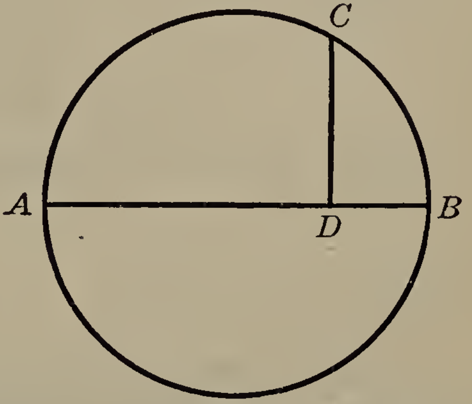
5. Find a mean proportional between $12\,mx^4$ and $3\,m^3$.

6. Find a fourth proportional to 1, $a + b$, and $a^2 - ab + b^2$.

7. Find a mean proportional to $(x^2 + ax + a^2)^2$ and $(x^2 - ax + a^2)^2$.

8. Find a third proportional to $(x - y)^2$ and $x^2 - y^2$.

9. In the adjacent figure CD is perpendicular to the diameter AB. It is a theorem of geometry that CD is a mean proportional between AD and DB. If the radius is 26 and BD is 8, find CD.



10. If four quantities are in proportion, and the third is a mean proportional between the first and second, prove that the second is a mean proportional between the third and fourth.

264. Test of a proportion. Since a proportion is an equality between two ratios (fractions), it is therefore an equation.

Hence any operation which may be performed on an equation may be performed on a proportion. (See Axioms, pages 56–58.)

Thus, in the proportion $\frac{a}{b} = \frac{c}{d}$ both members may be multiplied by bd, giving $ad = bc$. Here the first member is the product of the extremes of the proportion, and the second member is the product of the means.

Therefore in any proportion the product of the extremes equals the product of the means.

265. Proportions from equal products. The numbers which occur in a pair of equal products may be used in various ways as the terms of a proportion.

Thus, if $$ad = bc,$$

we may write either $$\frac{a}{b} = \frac{c}{d}, \text{ or } \frac{a}{c} = \frac{b}{d}.$$

Proof. If $a \cdot d = b \cdot c$ is divided by bd, we obtain

$$\frac{ad}{bd} = \frac{bc}{bd}, \text{ or } \frac{a}{b} = \frac{c}{d}. \qquad (1)$$

If $a \cdot d = b \cdot c$ is divided by cd, we obtain

$$\frac{a}{c} = \frac{b}{d}. \qquad (2)$$

If the means in (1) are interchanged, (2) is obtained. This process of obtaining (2) from (1) is called *alternation.*

If $a \cdot d = b \cdot c$ is divided by ac, we obtain

$$\frac{d}{c} = \frac{b}{a}. \qquad (3)$$

If the fractions in (1) are inverted, (3) is obtained. This process of obtaining (3) from (1) is called *inversion.*

EXERCISES

1. Write by alternation (*a*) $\frac{p}{q} = \frac{r}{s}$; (*b*) $\frac{7}{12} = \frac{10\frac{1}{2}}{18}$.

2. Write by inversion (*a*) $\frac{p}{q} = \frac{r}{s}$; (*b*) $\frac{13}{12} = \frac{17\frac{1}{3}}{16}$.

3. Form proportions from the following:

(*a*) $3 \cdot 8 = 2 \cdot 12$.
(*b*) $8 \cdot 7 = 14 \cdot 4$.
(*c*) $uv = xy$.
(*d*) $2x = 3m + 6$.
(*e*) $xy = a$.
(*f*) $a^2 - b^2 = x^2 - y^2$.

4. Find a mean proportional between:

(*a*) 1.44 and .0256;
(*b*) $\frac{1}{4}$ and 36;
(*c*) $a^2 - 2ab + b^2$ and m^2.

5. Find a fourth proportional to:

(*a*) 8, 6, and 30;
(*b*) 100, 64, and 27;
(*c*) x^2, x^3, and mx.

6. Find a third proportional to:

(*a*) 7 and 40;
(*b*) ax and x^3.

Solve for x:

7. $\frac{8}{10} = \frac{x}{5}$.

8. $\frac{12}{13} = \frac{5}{x}$.

9. $x:8 = 4:15$.

10. $7:34 = 9:x$.

11. $\frac{a+b}{a-b} = \frac{c}{x}$.

If $\frac{a}{b} = \frac{c}{d}$, prove the following and state the corresponding theorems in words:

12. $\frac{a}{c} = \frac{b}{d}$.

13. $\frac{b}{a} = \frac{d}{c}$.

14. $\dfrac{a^n}{b^n} = \dfrac{c^n}{d^n}$.

15. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \dfrac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

16. $\dfrac{a+b}{b} = \dfrac{c+d}{d}$.

HINT. Add 1 to each member of $\frac{a}{b} = \frac{c}{d}$.

17. $\dfrac{a+b}{a} = \dfrac{c+d}{c}$.

HINT. Write $\frac{a}{b} = \frac{c}{d}$ by inversion and add 1 to each member of the result.

18. $\dfrac{a-b}{b} = \dfrac{c-d}{d}$.

19. $\dfrac{a-b}{a} = \dfrac{c-d}{c}$.

20. $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.

The proportions given in Exercises 16, 18, and 20 are said to be derived from $a : b = c : d$ by *addition*, *subtraction*, and *addition and subtraction*, respectively.

If $\frac{a}{b} = \frac{c}{d}$, show that the following equalities are true:

21. $\dfrac{5a}{b} = \dfrac{5c}{d}$.

22. $\dfrac{2a}{7b} = \dfrac{2c}{7d}$.

23. $\dfrac{a^2}{b^2} = \dfrac{ac}{bd}$.

24. $\dfrac{5a+b}{5a-b} = \dfrac{5c+d}{5c-d}$.

25. $\dfrac{a^2-b^2}{b^2} = \dfrac{c^2-d^2}{d^2}$.

26. $\dfrac{a^2-7b^2}{b^2} = \dfrac{c^2-7d^2}{d^2}$.

27. $\dfrac{a^2-b^2}{2ab} = \dfrac{c^2-d^2}{2cd}$.

28. $\dfrac{7a^2-3b^2}{5ab} = \dfrac{7c^2-3d^2}{5cd}$.

29. $\dfrac{a^2+ab+b^2}{c^2+cd+d^2} = \dfrac{a^2-ab+b^2}{c^2-cd+d^2}$.

30. In the proof which follows give the reason for each step and state the result verbally.

If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, then $\dfrac{a+c+e}{b+d+f} = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$.

Proof. Setting each of the given ratios above equal to r,

$$\frac{a}{b} = r,\ \frac{c}{d} = r, \text{ and } \frac{e}{f} = r. \qquad (1)$$

Then from (1), $a = br,\ c = dr,\ e = fr.$ (2)

Adding in (2), $a + c + e = br + dr + fr.$ (3)

Factoring in (3), $a + c + e = (b + d + f)r.$ (4)

Therefore $$\frac{a + c + e}{b + d + f} = r. \qquad (5)$$

Hence by (1) and (5),

$$\frac{a + c + e}{b + d + f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

31. If $\frac{a}{b} = \frac{c}{d} = \frac{u}{v} = \frac{x}{y}$, prove $\frac{a + c + u + x}{b + d + v + y} = \frac{u}{v}$.

32. If $\frac{2}{3} = \frac{10}{15} = \frac{12}{18}$, show that $\frac{2 + 10 + 12}{3 + 15 + 18} = \frac{2}{3}$.

33. If $\frac{a}{b} = \frac{x}{y} = \frac{\sqrt{5}}{\sqrt{7}}$, prove that $\frac{a + x + \sqrt{5}}{b + y + \sqrt{7}} = \frac{a}{b}$.

Solve for x:

34. $(x + 3) : (x - 5) = 12 : 23.$

35. $(3x + 1) : (4x - 3) = 4x : (2x - 1).$

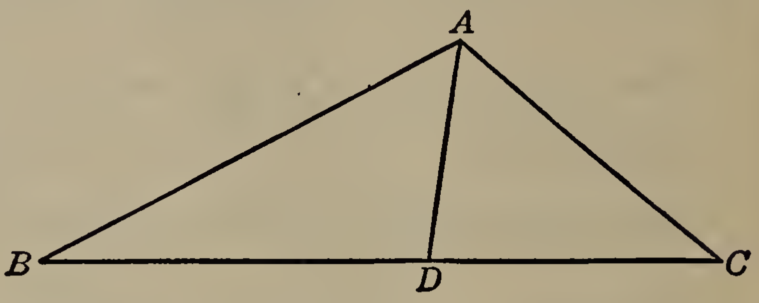
36. $(a + b) : (a - b) = (c + x) : (c - x).$

37. $(a + c + e) : (b + d + x) = a : b.$

PROBLEMS

1. A lamp post 9 feet high casts a shadow 7 feet long. How high is a flag pole which at the same time casts a shadow 42 feet long?

2. The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides. In triangle ABC if $AB = 18$, $AC = 12$, and $BC = 25$, find the segments of side BC made by the bisector of angle A.



3. The areas of two similar polygons are to each other as the squares of any two corresponding sides. If the corresponding sides of two similar polygons are 30 inches and 42 inches respectively and the area of the second is 1296 square inches, find the area of the first.

4. Two corresponding sides of two similar trapezoids are 18 feet and 28 feet respectively. The area of the first is 196 square feet. Find the area of the second.

5. The area of the surface of a sphere is $4\pi r^2$. If S represents the surface, R the radius, and D the diameter, show that for any two spheres

$$\frac{S_1}{S_2} = \frac{R_1^2}{R_2^2} = \frac{D_1^2}{D_2^2}.$$

6. Compare the areas of two spheres if their radii have the ratio $3:5$.

7. The diameter of the moon is 2160 miles and of the earth 7920 miles. Find the ratio of their surfaces.

8. The volume of a sphere is $\frac{4\pi R^3}{3}$. If V represents the volume, show that for any two spheres

$$\frac{V_1}{V_2} = \frac{R_1^3}{R_2^3} = \frac{D_1^3}{D_2^3}.$$

9. Compare the volumes of two spheres whose radii are in the ratio 5 : 7.

10. Combining the results of Problems 5 and 8 show that

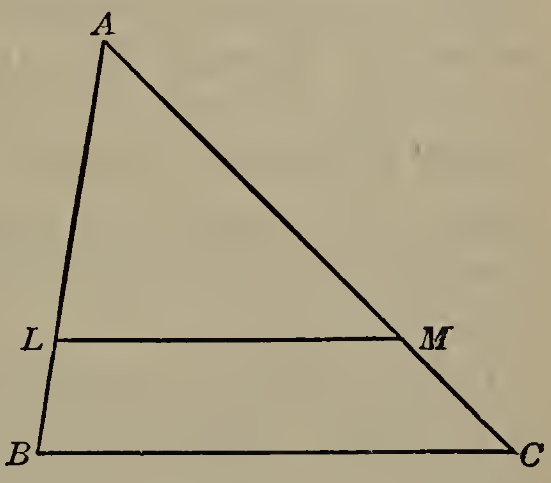
$$\frac{V_1}{V_2}=\frac{S_1^{\frac{3}{2}}}{S_2^{\frac{3}{2}}}.$$

11. The radius of the sun is to the radius of the earth as 109 : 1. Find the ratio of their volumes and the ratio of their surfaces.

12. If *ABC* is any triangle and *LM* is a line parallel to *BC*, meeting *AB* at *L* and *AC* at *M*, it is proved in geometry that

$$\frac{\text{area } ABC}{\text{area } ALM}=\frac{\overline{AB}^2}{\overline{AL}^2}=\frac{\overline{AC}^2}{\overline{AM}^2}=\frac{\overline{BC}^2}{\overline{LM}^2}.$$

If in the adjacent figure the area of triangle *ABC* is 196 square inches and *AL* is 15 inches and *AB* is 21 inches, find the area of the triangle *ALM*. What is the area of the trapezoid *BCML*?



13. If in the figure of Exercise 12, *AL* is 18 and *AB* is 28 and *ALM* equals 300, find the area of *ABC*.

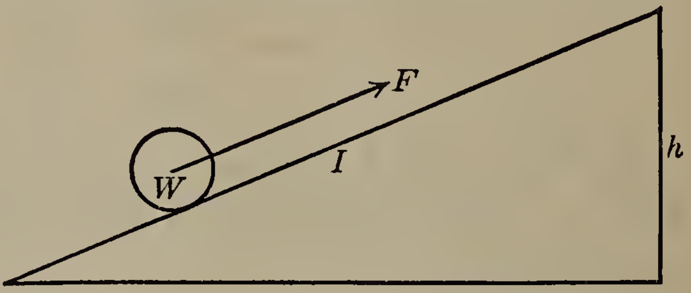
14. If in the figure of Exercise 12, *AL* is 12 feet and *AB* is 20 feet, and area *ABC* equals 400 square feet, find the area of the trapezoid *BCML*.

15. If in the figure of Exercise 12, *AL* is 60 feet and area *ALM* : area *BCML* = 4 : 5, find *AB*.

16. If in the figure of Exercise 12, *ABC* equals twice *BCML* and *AB* = 100, find *AL* to two decimals.

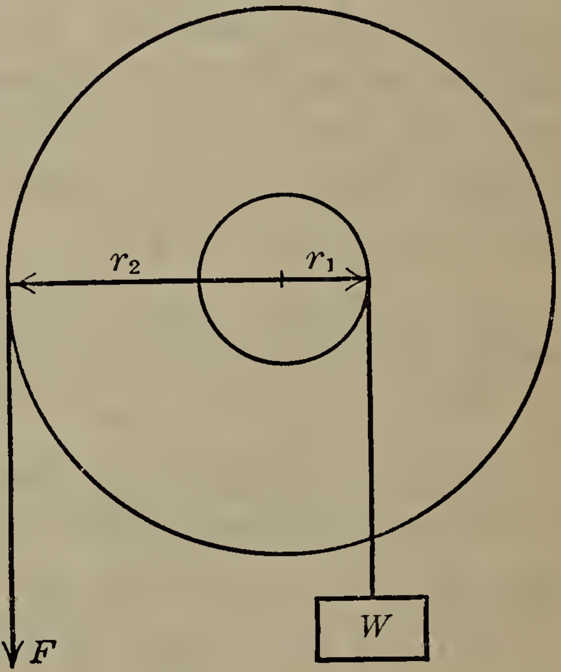
17. Compare the radii of two spheres whose volumes are in the ratio of 64 : 343.

18. It is shown in physics that, ignoring friction, the force F required to hold a weight W on the inclined plane of length I and height h is determined by the proportion $F : W = h : I$. If h is 3 feet and I is 15 feet, what force will be required to hold a weight of 500 pounds?



19. If W is 240 pounds and the ball-bearing carriage carrying it weighs 30 pounds, what force will hold W if h is 5 and $I = 13$?

20. The adjacent figure shows a wheel and axle. It is proved in physics that the force F applied to the circumference of the wheel of radius r_2 will just balance a weight W hung by a rope around the cylinder of radius r_1, provided $F : W = r_1 : r_2$. If the radius of the wheel is 10 inches and that of the axle is 3 inches, what force will be required to raise a weight of 200 pounds?



21. The crank of a windlass is 24 inches long and the cylinder on which the rope carrying the bucket winds up is 8 inches in diameter. Find the force on the crank required to lift a bucket of rock weighing 280 pounds.

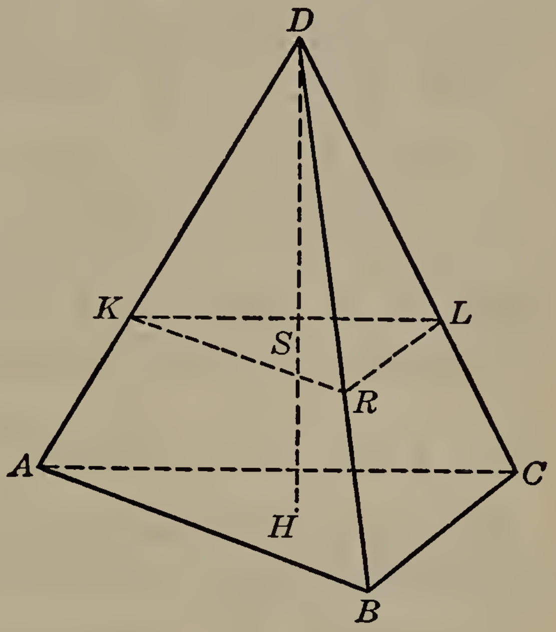
22. If a plane be passed parallel to the base of a pyramid as in the accompanying figure, cutting it in *KRL*, then

$$\frac{\text{pyramid } DABC}{\text{pyramid } DKRL} = \frac{\overline{DH}^3}{\overline{DS}^3} = \frac{\overline{DK}^3}{\overline{DA}^3}, \text{ etc.}$$

If in the adjacent figure the pyramids are 64 and 125 cubic inches respectively and the altitude *DH* is 50 inches, find *DS*.

23. If the frustum *ABCKRL* is $\frac{26}{27}$ of the whole pyramid and $DS = 24$, find *DH*.

24. If *KRL* divides the pyramid into two equal parts and $DH = 10$, find *DS* and *SH* to two decimals.



266. Variation. The word *quantity* denotes anything which is *measurable,* such as distance, rate, time, and area.

Many operations and problems in mathematics deal with numerical measures of quantities, some of which are fixed and others constantly changing. Such problems as deal with the relation of the numeric measures of at least two changing quantities are called problems in variation.

The theory of variation is really involved in proportion. This fact will become evident after a study of the illustrations of the different types of variation here given.

The equation $x = 5y$ may refer to no physical quantities whatever, yet it is possible to imagine y as taking on in succession every possible numeric value, and the value of x as changing with every change of y, and consequently always being five times as great as the corresponding value of y. In this sense, which is strictly mathematical, x and y are variables.

267. Direct variation. One hundred feet of iron wire of a certain size weighs 8 pounds. Obviously 200 feet of the same wire would weigh 16 pounds, a piece 300 feet long would weigh 24 pounds, and so on.

Here we have two variables, W(weight) and L(length), so related that the value of W depends on the value of L, in such a way that W increases proportionately as L increases. That is, W is directly proportional to L. Hence if W_1 and W_2 are any two weights corresponding to the lengths L_1 and L_2 respectively

$$W_1 : W_2 = L_1 : L_2. \qquad (1)$$

The fact expressed by (1) can be stated in the form of a variation, thus: W varies as L.

In general, if x varies as y, and x and y denote *any* two corresponding values of the variables, and x_1 and y_1 a *particular* pair of corresponding values of these variables,

then $$\frac{x}{x_1} = \frac{y}{y_1}. \qquad (2)$$

From (2), $$x = \left(\frac{x_1}{y_1}\right)y. \qquad (3)$$

But $\frac{x_1}{y_1}$ is a constant, being the quotient of two definite numbers, while x and y denote any corresponding values of the variables.

Call this constant K, and (3) may be written

$$x = Ky. \qquad (4)$$

That is, *if one variable varies as a second, the first always equals the second multiplied by some constant.*

Thus for the iron wire just mentioned, $W = \frac{8}{100} L$, or $\frac{2}{25} L$. Here, though W varies as L varies, W is always equal to L multiplied by the constant $\frac{2}{25}$.

EXERCISES

1. If x varies as y and $x = 5$ when $y = 9$, find y when x is 16.

Solution. The variation is direct.

Therefore, $$\frac{5}{16} = \frac{9}{y}.$$

Solving, $$y = 28\tfrac{4}{5}.$$

2. If x varies as y and $x = 12$ when $y = 35$, find y when $x = 27$.

3. If x varies as y and $x = n$ when $y = v$, find x when y is m.

4. If x varies as y and $x = 6$ when $y = 15$, find k in $x = ky$.

5. A certain quantity of gas is inclosed in a gas holder. If the volume of the gas varies as the temperature and the volume is 500 cubic feet when the temperature (absolute) is 300°, what is the volume when the temperature rises 30°? when it falls 100°?

6. If the volume of gas in a gas holder is 1,000,000 cubic feet at 300°, find the change in temperature if it causes an increase in volume of 100,000 cubic feet.

7. One quantity may vary directly as the square root of another. Express this relation in symbols: (*a*) as a variation, (*b*) as a proportion.

8. One quantity may vary as the cube of another. Express this relation in symbols: (*a*) as a variation, (*b*) as a proportion.

268. Inverse variation. If two variables are so related that one increases as the other decreases and vice versa, either variable is said to vary inversely as the other. For example, if the area of a triangle, A, is constant while the base, b, and altitude, a, vary, the base varies inversely as the altitude. This is apparent from the equation $A = \frac{1}{2}ab$.

Similarly if d is the distance covered by a motor bus each trip, and r its speed, and t the time, $d = rt$. Here for a fixed route of length d, t varies inversely as r.

Therefore
$$t_1 : t_2 = \frac{1}{r_1} : \frac{1}{r_2}. \qquad (5)$$

In the form of a variation (5) becomes t varies as $\frac{1}{r}$.

In general, x varies *inversely as* y when x varies as the *reciprocal of* y; that is,

$$x \text{ varies as } \frac{1}{y}. \qquad (6)$$

And if x and y denote any two corresponding values of the variable, and x_1 and y_1 a particular pair of corresponding values,

$$x : x_1 = \frac{1}{y} : \frac{1}{y_1}. \qquad (7)$$

Whence
$$\frac{x}{y_1} = \frac{x_1}{y}, \text{ or } xy = x_1y_1. \qquad (8)$$

But x_1y_1 is a constant, being the product of two definite numbers. Call this constant K.

Then (8) becomes $xy = K$.

That is, if one variable varies inversely as another, the product of the two is a constant.

Many phenomena in nature are governed by a law which involves inverse variation, and the square or the cube of one of the variables concerned. Thus the force of gravitation between two bodies varies inversely as the square of the distance between them. And the force with which a magnet attracts a piece of soft iron varies inversely as the square of the distance between them. Also the intensity of light or sound varies inversely as the square of the distance of the observer from the source of the light or the sound. The tide-producing force of the moon on the ocean varies inversely as the cube of the moon's distance from the earth.

EXERCISES

1. If x varies inversely as t and $x = 10$ when t is 2, find x when t is 60.

Solution. The variation is inverse.

Therefore, $$10 : x = \tfrac{1}{2} : \tfrac{1}{60}.$$

Solving, $$x = \tfrac{1}{3}.$$

2. If x varies as $\dfrac{1}{y}$ and $x = 3$ when $y = 8$, find x when y is 4.

3. If y varies as $\dfrac{1}{t}$ and $y = u$ when $t = x$, find y when $t = m$.

4. If s varies as $\dfrac{1}{d}$ and $s = 4$ when $d = 6$, find d when $s = 20$.

5. If the pressure on a certain quantity of gas in a gas holder is changed without changing the temperature, then Boyle's law holds. This is sometimes stated $p_1v_1 = p_2v_2$, where p represents pressure and v the volume. State this equation as a proportion. State it as a variation.

6. Six hundred cubic feet of gas in a gas holder is under a pressure of 24 pounds to the square inch. Use Boyle's law and determine the volume if without changing the temperature the pressure is changed to (*a*) 20 pounds; (*b*) 30 pounds.

7. The pressure on 3600 cubic feet of air at 60 pounds per square inch is changed so that the volume becomes 2400 cubic feet. Find the pressure.

8. One quantity may vary inversely as the square root of another. Express such a relation in symbols (*a*) as a variation, (*b*) as a proportion.

9. One quantity may vary inversely as the square or the cube of another. Express each of these relations in symbols (*a*) as a proportion, (*b*) as a variation.

269. Joint variation. One quantity is said to vary jointly as two others when it varies directly as their product. For example, the wages of a workman varies jointly with the amount he receives per hour and the number of hours he works. If W represents his total wages, p the amount received per hour, and n the number of hours he works, $W = np$. Similarly the formula for the area of a rectangle $A = ab$ is an example of joint variation. One variable often varies jointly as the product of more than two variables.

Thus the pressure of a horizontal wind on a roof varies jointly as the square of the wind's velocity, the area of the roof, and the cosine of the angle which the roof makes with the vertical.

In general, any variable x varies jointly as two others, y and z, if

$$x \text{ varies as } yz; \qquad (1)$$

that is, if x varies as the product of the two.

If x varies jointly as y and z, and if x, y, and z denote any corresponding values of the variables, while x_1, y_1, and z_1 denote a particular set of such values, then

$$\frac{x}{x_1} = \frac{yz}{y_1z_1}. \tag{2}$$

From (2),
$$x = \left(\frac{x_1}{y_1z_1}\right) yz. \tag{3}$$

But in (3) the fraction $\frac{x_1}{y_1z_1}$ is a constant, since x_1, y_1, and z_1 are particular values of the variables x, y, and z. Calling this constant K, we may write x varies as yz as the equation

$$x = Kyz.$$

One variable may vary directly as another variable (or several variables) and inversely as still another (or several others). Also one variable may vary as the square, or the cube, or the square root, or the reciprocal, or as any algebraic expression whatever involving the other variable (or variables).

EXERCISES

1. If x varies jointly as y and z, and $x = 30$ when $y = 8$ and $z = 10$, find x when $y = 18$ and $z = 4$.

Solution. The variation is joint.

Therefore,
$$\frac{30}{x} = \frac{8 \cdot 10}{18 \cdot 4}.$$

Solving,
$$x = 27.$$

2. If x varies as uv, and $x = 300$ when $u = 35$ and $v = 24$, find x when $u = 100$ and $v = 60$.

3. If A varies as hb, and $A = 84$ when $h = 16$ and $b = 50$, find A when $h = 30$ and $b = 40$.

PROBLEMS

1. The intensity (brightness) of light varies inversely as the square of the distance from the source of light. A book is held 2 feet from a lamp and later 5 feet distant. At which distance does it appear brighter? How many times as bright?

2. The intensity of sound varies inversely as the square of the distance from the source of the sound. Compare the intensity of the sound of a bell for a listener who is at first 300 feet away and later is 2 miles distant.

3. The safe load, W, for a beam varies jointly as the thickness t, the depth d, and inversely as the length l between the supports. Express this relation between W, t, d, and l by a proportion.

4. A safe load of 2000 pounds is carried by a beam 16 feet long, 4 inches thick, and 18 inches deep. What is the safe load for a beam of the same material half as long, half as thick, and half as deep?

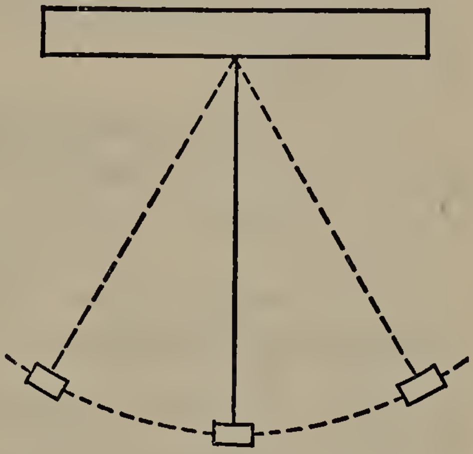
5. If x varies as y and inversely as z, represent this relation in symbols by a proportion.

6. If x varies as y^2 and inversely as the square root of z, represent this relation in symbols by a proportion and by a variation.

7. The volume of a given weight of gas varies inversely as the pressure and directly as the temperature. Express this relation between v, p, and t by a proportion and by a variation.

8. If the volume is 600 when $p = 32$ and $t = 320$, find the volume when $p = 48$ and $t = 360$.

9. The time of vibration of a pendulum varies directly as the square root of its length and inversely as the force of gravitation. Express the relation between t, l, and g by a proportion.



10. If a pendulum 100 centimeters long vibrates once in 1 second, find the period of a pendulum 121 centimeters long.

11. If a pendulum 100 centimeters long vibrates once in one second at sea level where the force of gravitation is 982, find the time of one vibration of a pendulum 144 centimeters long on top of a mountain where the force of gravitation is 970.

12. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. Express this relation as a variation and also as a proportion.

13. An object weighing 160 pounds at the earth's surface would weigh how much 800 miles above the surface? 2000 miles above the earth's surface? (Radius of earth equals 4000 miles.)

14. A meteorite weighs one ton. How far from the earth's surface was it when its weight was half as much? one tenth as much?

15. The weight of any object below the surface of the earth varies directly as its distance from the center. An object weighs 100 pounds at the earth's surface. Find its weight (*a*) 800 miles below the surface, (*b*) 2000 miles below the surface, (*c*) at the center. (Radius of earth = 4000 miles.)

16. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. Express this relation as a variation and as a proportion.

17. If the pressure of the wind blowing 20 miles per hour is .8 of a pound on one square foot, find the pressure on one wall of a building 100 feet long and 120 feet high.

18. Newton's law of gravitation states that the force of attraction between two bodies (spheres) varies as the product of their masses and inversely as the square of the distance between them. Express this relation as a variation and as a proportion.

19. The mass of the sun is 330,000 times the mass of the earth and the distance between them (that is, the distance between their centers) is 93,000,000 miles. The force of attraction between the sun and the earth is 36×10^{17} tons. If the earth's mass is 81 times the mass of the moon, find the attraction between the earth and the moon if the distance between them is 240,000 miles.

CHAPTER XXXIX

DETERMINANTS

270. Determinants of the second order. The arrangement of numbers $\begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix}$ has been given the meaning $5 \cdot 3 - 4 \cdot 2$, which equals 7.

Such an arrangement is called a *determinant.* Since it has two rows and two columns, it is said to be of the *second order.*

A determinant of the second order really represents a number, and this number is easily found since

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Accordingly $\begin{vmatrix} 7 & 3 \\ 6 & 4 \end{vmatrix} = 7 \cdot 4 - 3 \cdot 6 = 28 - 18 = 10.$

Similarly $\begin{vmatrix} 5 & -6 \\ 2 & 3 \end{vmatrix} = 5 \cdot 3 - 2(-6) = 15 + 12 = 27.$

The preceding operations can be reversed and the difference or sum of two products written as a determinant of the second order.

Thus $pq - uv$ can be written $\begin{vmatrix} p & v \\ u & q \end{vmatrix}$ or $\begin{vmatrix} p & u \\ v & q \end{vmatrix}$ or $\begin{vmatrix} q & v \\ u & p \end{vmatrix}$ or $\begin{vmatrix} q & u \\ v & p \end{vmatrix}$ since each equals $pq - uv$.

Similarly $mn - r = m \cdot n - 1 \cdot r = \begin{vmatrix} m & r \\ 1 & n \end{vmatrix}.$

Also $3 \cdot 7 - 2 \cdot 4 = \begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix}.$

EXERCISES

Find the value of the determinants:

1. $\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}.$

2. $\begin{vmatrix} 5 & -2 \\ 3 & 4 \end{vmatrix}.$

3. $\begin{vmatrix} 11 & 9 \\ 4 & 7 \end{vmatrix}.$

4. $\begin{vmatrix} 8 & -3 \\ 2 & 5 \end{vmatrix}.$

5. $\begin{vmatrix} 6 & 5 \\ 7 & 3 \end{vmatrix}.$

6. $\begin{vmatrix} 18 & 4 \\ 12 & 3 \end{vmatrix}.$

7. $3\begin{vmatrix} 15 & 4 \\ 18 & -3 \end{vmatrix}.$

8. $4a\begin{vmatrix} -9 & 68 \\ 14 & 16 \end{vmatrix}.$

9. $m\begin{vmatrix} 15 & 11 \\ -6 & 8 \end{vmatrix}.$

10. $\frac{7}{3m}\begin{vmatrix} m^3 & 0 \\ m^2 & 2m \end{vmatrix}.$

11. $ax\begin{vmatrix} (a-x) & a \\ (a+x) & x \end{vmatrix}.$

12. $\frac{-4}{3}\begin{vmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{vmatrix}.$

Write as a determinant:

13. $mx - ny.$

14. $an - bc.$

15. $az + xy.$

16. $ab - m.$

17. $cd - ab.$

18. $x - y.$

Write as the quotient of two determinants:

19. $\frac{ab - cd}{pq - ml}.$

20. $\frac{am - 15}{2x - 2y}.$

21. $\frac{\frac{x}{2} - 6}{3a - 1}.$

22. $\frac{5a - xy}{4x - 3a}.$

23. $\frac{x^2 - y^2}{x^2 + y^2}.$

24. $\frac{ax + ay}{bc - ad}.$

271. Solution by determinants. The foregoing principles are easily applied to the solution of any linear system in two unknowns by means of determinants of the second order.

The ordinary solution of the system

$$ax + by = c \qquad (1)$$

$$dx + ey = f \qquad (2)$$

gives $x = \dfrac{ce - bf}{ae - bd}$ and $y = \dfrac{af - cd}{ae - bd}$.

Using determinants

$$x = \frac{ce - bf}{ae - bd} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad (3)$$

and

$$y = \frac{af - cd}{ae - bd} = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}. \qquad (4)$$

If the denominators in (3) and (4) are zero and the numerators are not zero, then no set of roots exists for (1) and (2).

The determinant expressions for x and y in (3) and (4) can be used as general formulas to solve any pair of linear equations in two unknowns. The determinant forms can be easily remembered and written down at once if we observe the following points:

I. *The determinants in the denominators are identical, and each is formed by the coefficients of* x *and* y *as they stand in the original equations* (1) *and* (2).

II. *The determinant in the numerator of the value of* x *is formed from the denominator by replacing the coefficients of* x, $\begin{matrix} a \\ d \end{matrix}$, *by the constant terms* $\begin{matrix} c \\ f \end{matrix}$.

III. *The determinant in the numerator of the value of* y *is formed from the denominator by replacing the coefficients of* y, $\begin{smallmatrix} b \\ e \end{smallmatrix}$, *by the constant terms* $\begin{smallmatrix} c \\ f \end{smallmatrix}$.

EXAMPLE

Solve by determinants: $\begin{cases} 3y + 13x = 14, \\ 7x - 2y = 22. \end{cases}$

Solution. Writing the equations in standard form, we have

$$\begin{cases} 13x + 3y = 14, \\ 7x - 2y = 22. \end{cases}$$

Then $$x = \frac{\begin{vmatrix} 14 & 3 \\ 22 & -2 \end{vmatrix}}{\begin{vmatrix} 13 & 3 \\ 7 & -2 \end{vmatrix}} = \frac{-28-66}{-26-21} = \frac{-94}{-47} = 2.$$

In solving for y the denominator is the same as before:

Hence $$y = \frac{\begin{vmatrix} 13 & 14 \\ 7 & 22 \end{vmatrix}}{-47} = \frac{286-98}{-47} = \frac{188}{-47} = -4.$$

Check as usual.

EXERCISES

Solve by determinants and check results:

1. $5x + 2y = 16,$
$3x + 4y = 18.$

2. $4x + y = 2,$
$x - 2y = 5.$

3. $7h - k = 2,$
$6h - k = 3.$

4. $3x = 4y + 14,$
$3y = 4x - 14.$

5. $6s + 8t = 26,$
$5s - 3t = 70.$

6. $4x + \frac{2z}{3} = \frac{26}{3},$
$3z + \frac{7x}{2} + 4 = 0.$

7. $\frac{5x}{6} + \frac{y}{4} = 7,$
$\frac{2x}{3} - \frac{y}{8} = 3.$

8. $4x + 2y = m,$
$2x + 4y = n.$

9. $my + 3z = 3 + 6m,$
$m^2y + z = 5m.$

10. $au + bv = a + b,$
$au - bv = a - b.$

11. $\frac{u}{m} + \frac{v}{n} = \frac{m+n}{mn},$
$u - v = \frac{m^2 - n^2}{mn}.$

12. $ay + (b+c)z = 1,$
$az = 1 - (b+c)y.$

272. Determinants of the third order. By definition any arrangement of numbers like:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1. \qquad (1)$$

Such an arrangement is called a determinant of the *third order* because it has three *rows* and three columns. Each of the nine numbers in the determinant is called an *element.*

Grouping terms in the right member of (1), we get

$$(a_1b_2c_3 - a_1b_3c_2) + (a_2b_3c_1 - a_2b_1c_3) + (a_3b_1c_2 - a_3b_2c_1), \qquad (2)$$

or $$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1), \qquad (3)$$

and $$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \qquad (4)$$

From (1), (2), (3), and (4)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \quad (5)$$

Equation (5) shows that a determinant of the third order can be expressed in terms of three determinants of the second order, the elements of the latter being elements of the third-order determinant.

The second-order determinants of (5) can always be easily written because their method of formation is governed by two principles:

I. *The coefficients of the second-order determinants are the elements of the first column of the third-order determinant, the sign of* a_2 *being minus.*

II. *The elements of the second-order determinants are those* not *in the same row or column with the coefficients* a_1, a_2, a_3. *Thus the elements of* $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ *are not in the first row or column as is the coefficient* a_1.

Hence $$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 1 & 6 \\ 3 & 7 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 6 \\ 7 & 4 \end{vmatrix} - 5\begin{vmatrix} 3 & 4 \\ 7 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix}$$

$$= -76 + 80 + 42 = 46.$$

EXERCISES

Find the value of:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 5 & 3 \end{vmatrix}.$ **2.** $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 1 \end{vmatrix}.$ **3.** $\begin{vmatrix} 5 & -1 & 1 \\ 6 & -2 & 0 \\ 0 & 2 & 3 \end{vmatrix}.$

4. $$12\begin{vmatrix} 3 & -2 & 4 \\ 2 & 3 & -2 \\ 5 & 2 & 3 \end{vmatrix}.$$

5. $$\begin{vmatrix} 9 & 5 & 4 \\ 5 & 6 & 5 \\ 16 & 4 & 6 \end{vmatrix}.$$

6. $$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 8 \end{vmatrix}.$$

7. $$\begin{vmatrix} 5 & -7 & 4 \\ 6 & 3 & -5 \\ 4 & 6 & 3 \end{vmatrix}.$$

8. $$\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ -8 & 6 & 1 \end{vmatrix}.$$

9. $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

10. $$\begin{vmatrix} a & 3 & 4 \\ b & 4 & 5 \\ c & 5 & 6 \end{vmatrix}.$$

11. $$bc\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}.$$

12. $$\begin{vmatrix} (a+b) & 2 & 1 \\ 2 & (a-b) & 1 \\ 3 & 3 & 1 \end{vmatrix}.$$

273. The general linear system in three variables. The method of addition and subtraction applied to the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \\ a_2x + b_2y + c_2z = d_2, & (2) \\ a_3x + b_3y + c_3z = d_3, & (3) \end{cases}$$

gives $$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \qquad (4)$$

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \qquad (5)$$

$$z = \frac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \qquad (6)$$

By grouping and factoring as in section 272 the values (4), (5), and (6) become

$$(7)\quad x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\qquad (8)\quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\qquad (9)\quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

The fractions (4), (5), and (6) are general results, and can be used as formulas to solve any three simultaneous equations in three variables, but the equivalent forms (7), (8), and (9) are far more easily remembered. These can be written down at once for any system of three equations in three variables since

I. *The determinants in the denominators are identical and each is formed by the coefficients of* x, y, z, *as they stand in the original equations.*

II. *Each determinant in the numerator is formed from the denominator by putting the column of constant terms (as they stand in the original system) in place of the column of the coefficients of the variable whose value is sought.*

If the denominator of (7), (8), and (9) is zero and the numerators are not zero, then no set of roots exist for (1), (2), and (3).

EXAMPLE

Solve, using determinants:

1.
$$\begin{cases} x + 3y = 2 - 5z, & (1) \\ 2x + z = y + 1, & (2) \\ 3x + 5y + 7z = -10. & (3) \end{cases}$$

Solution. Rewriting in standard form,

$$x + 3y + 5z = 2, \tag{4}$$

$$2x - y + z = 1, \tag{5}$$

$$3x + 5y + 7z = -10. \tag{6}$$

From I and II, page 616,

$$x = \frac{\begin{vmatrix} 2 & 3 & 5 \\ 1 & -1 & 1 \\ -10 & 5 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 5 \\ 2 & -1 & 1 \\ 3 & 5 & 7 \end{vmatrix}} = -\frac{100}{20} = -5$$

$$y = \frac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 1 \\ 3 & -10 & 7 \end{vmatrix}}{20} = -\frac{120}{20} = -6.$$

The value of z can now be more easily obtained by substituting the values of x and y already found in (1), (2), or (3) than by determinants.

Substituting in (2), $-10 + z = -6 + 1.$

Hence, $z = 5.$

Check as usual.

EXERCISES

Solve for x, y, and z as in the preceding example and check results.

1. $x + y + z = 27,$
   $x - y + z = 9,$
   $x + y - z = 5.$

2. $2x + y - z = 3,$
   $x - 3y + z = 2,$
   $x + y - 5z = -8.$

3. $3x + y = 22,$
   $x + y - 3z = -7,$
   $x + 3z + y = -11.$

HINT. The first equation is $3x + y + 0z = 22$. The coefficient zero must never be omitted in using determinants.

4. $3x + y = 25,$
$x + 2y = 5,$
$2y + z = 2.$

5. $x + y = 2a,$
$x + z = 3a,$
$y + z = 4a.$

6. $.4x + .3y - 8z = 4,$
$.5x + z + .8y = 1.2,$
$2.6z + .3 - x = .5y.$

7. $3x + 2z = 6a - 2y,$
$x - 5y + 6z = 2a - 11b,$
$6x - 8z = 12a + 8b.$

8. $ax + by = 0,$
$cx - by = 2bc,$
$bx + az - cy = b^2.$

9. $ax + by - cz = 2ab,$
$ay - bx + cz = 2bc,$
$ax - by + cz = 2ac.$

274. Determinants of the fourth order. Linear systems in four unknowns lead to determinants of the fourth order, linear systems in five unknowns lead to determinants of the fifth order, and so on. To find the value of a fourth-order determinant we first express it as four determinants of the third order. Each of these can then be evaluated as in section 273.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} +$$

$$a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

Note that the signs of the coefficients are alternately plus and minus. Similarly a fifth-order determinant can be expressed in terms of five determinants of the fourth order and so on. In general a determinant of order n can be expressed in terms of n determinants of order $n - 1$.

EXERCISES

Find the value of:

1. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 5 & 1 \end{vmatrix}.$

2. $\begin{vmatrix} 1 & 4 & 3 & 8 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 1 & 1 \end{vmatrix}.$

3. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 5 \end{vmatrix}.$

4. $\begin{vmatrix} x & y & z & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix}.$

5. $\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 4 & -3 & -1 & 1 \\ -4 & -1 & -1 & 1 \end{vmatrix}.$

6. $\begin{vmatrix} x & y & z & 1 \\ 10 & 10 & 1 & 1 \\ -8 & -8 & -8 & 1 \\ 6 & 0 & 0 & 1 \end{vmatrix}.$

NOTE. Like so many other discoveries, the determinant notation was noticed independently by two men. In a letter to a friend, writing in 1693, Leibnitz outlined the method of solving equations by means of determinants, but, so far as we know, he used the notation in his own work very little, and certainly did not publish it during his lifetime. In fact, the letter in which this reference is found, did not come to light until 1850, and the fact that Leibnitz knew anything about determinants was not generally recognized until after that time.

In 1750 Cramer, a professor in the university at Geneva, rediscovered this method of solving linear systems, and his work had the good fortune to be accepted by scholars, forming the real beginning of the development of the subject. Since that time a great many have written on the subject, and to-day determinants are used in every field of advanced mathematics.

c

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	**51**	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	**52**	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	**53**	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	**54**	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	**55**	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	**56**	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	**57**	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	**58**	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	**59**	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	**61**	3,721	226,981	7.810	3.937
12	144	1,728	3.464	2.289	**62**	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	**63**	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	**64**	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	**65**	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	**66**	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	**67**	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	**68**	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	**69**	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	**71**	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	**72**	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	**73**	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.885	**74**	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	**75**	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.963	**76**	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	**77**	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	**78**	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	**79**	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	**81**	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	**82**	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	**83**	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	**84**	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	**85**	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	**86**	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	**87**	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	**88**	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	**89**	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	**90**	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	**91**	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	**92**	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	**93**	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	**94**	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	**95**	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	**96**	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	**97**	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	**98**	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	**99**	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.642

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